Master-slave synchronization criteria for horizontal platform systems using time delay feedback control

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Abstract

This paper is concerned with master-slave synchronization for two identical non-autonomous horizontal platform systems by using time-delay feedback control. Compared with some existing results on synchronization for horizontal platform systems, the effect of the time-delay in the feedback control on master-slave synchronization is investigated. Applying a delay decomposition approach, some delay-dependent synchronization criteria are established and formulated in the form of linear matrix inequalities (LMIs). Sufficient conditions about the existence of a time delay feedback controller are derived by employing these newly-obtained synchronization criteria. The controller gains can be achieved by solving a set of LMIs. One simulation example is given to illustrate the effectiveness of synchronization criteria and the design

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method.

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1. Introduction

Chaos synchronization has received much attention due to its theoretical importance and practical applications, see for example, [1], [2], [3] and references therein. As is well known, horizontal platform systems are widely used in offshore engineering and earthquake engineering (see examples, [4] and [5]). Observing this point, Ge *et al.* (2003) [4] numerically investigated the synchronization problem for two horizontal platform systems. Cai *et al.* (2007) [6] used a sinusoidal state error feedback control and applied the Lyapunov direct method to derive some synchronization criteria for horizontal platform systems. Wu *et al.* (2006) [7] studied the master-slave synchronization problem for two horizontal platform systems by using linear feedback control. Wu *et al.* (2007) [8] considered the synchronization problem for two horizontal platform systems with phase difference. The idea in [6, 7, 8] is to transform the nonlinear error system into a linear time-varying system, and then to discuss the asymptotic stability of the linear time-varying system. The sufficient criteria derived by this method seem simple and are easily checked. However, noticing that the horizontal platform systems are essentially a class of nonlinear systems, the synchronization criteria in [6, 7, 8] may be conservative. How to make good use of the property of the nonlinear term of the error system to derive some less conservative synchronization criteria is the first motivation of the current study.

Due to the propagation delay frequently encountered in the master-slave synchronization scheme, recently, some researchers have made efforts to investigate the effect of a time-delay on master-slave synchronization. For example, Yalcin *et al.* (2001) [3] derived some delay-independent and delay-dependent synchronization criteria. Han (2007) [2] studied how to design time-varying delay feedback controllers for master-slave synchronization of a class of nonlinear systems. In a master-slave synchronization scheme for horizontal platform systems, there inevitably exist time-delays during the signal transferring from the master system to the controller and from the slave system to the controller. However, linear or sinusoidal state error feedback controllers were chosen to achieve the synchronization for horizontal platform systems in [4, 6, 7, 8], which did not consider the effect of the time-delay on the synchronization. If a time-delay occurs in the master-slave scheme, the method in [6, 7, 8] will transform the corresponding error system into a linear time-vary system with a time-delay. However, the derived synchronization criteria and feedback gains are not as simple as those corresponding results in [6, 7, 8]. To sum up, we are in a position to study the effect of the time-delay on the master-slave synchronization for horizontal platform systems and to design an effective time-delayed feedback controller, which is the second motivation of the current study.

In this paper, we will deal with the problem of master-slave synchronization of two identical horizontal platform systems by using time delay feedback control. Compered with some existing results, we will investigate the effect of the time-delay on the master-slave synchronization for horizontal platform systems. We will employ a delay decomposition approach recently proposed in [9], [10], [11], [12] and fully use information from the nonlinear term of the error system to derive the synchronization criteria. Based on the synchronization criteria, we will give some sufficient conditions on the existence of a state error feedback controller. These sufficient conditions will be formulated in the form of linear matrix inequalities (LMIs). Moreover, we will design the controller by solving a set of LMIs. We will use one simulation example to illustrate the effectiveness of synchronization criteria and the design method.

Notation: In this paper, all matrices are real matrices. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. For symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$, the notation $\mathbf{P} > \mathbf{Q}$ (respectively, $\mathbf{P} \geq \mathbf{Q}$) means that matrix $\mathbf{P} - \mathbf{Q}$ is positive definite (respectively, positive semi-definite). $\mathbf{I}_N$ is an identity matrix of $N \times N$ dimensions. $tr(\mathbf{W})$ denotes the trace of matrix W. $diag(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ denotes the diagonal matrix. For an arbitrary matrix $\mathbf{W}$ and two symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$, the symmetric term in a symmetric matrix is denoted by $*$, i.e. $\begin{pmatrix} \mathbf{P} & \mathbf{W} \\ * & \mathbf{Q} \end{pmatrix} = \begin{pmatrix} \mathbf{P} & \mathbf{W} \\ \mathbf{W}^T & \mathbf{Q} \end{pmatrix}$.

2. Problem Statement

Consider a horizontal platform system which is depicted in Fig. 1 [4, 5]. The mathematical model of the system can be described by

$$A\ddot{x} + D\dot{x} + rg\sin x - \frac{3g}{R_0}(B - C)\cos x \sin x = F\cos\omega t, \tag{1}$$

where x is the rotation of the platform relative to the earth; A, B and C are the inertia moment of the platform for axis 1, axis 2 and axis 3, respectively; D is the damping coefficient; r is the proportional constant of the accelerometer; g is the acceleration constant of gravity; R_0 is the radius of the earth; and $F\cos\omega t$ is the harmonic torque.

Let $y_1(t) = x(t)$ and $y_2(t) = \dot{y}_1(t) = \dot{x}(t)$. The horizontal platform system (1) can be rewritten as

$$\begin{cases} \dot{y}_1(t) = y_2(t), \\ \dot{y}_2(t) = -ay_2(t) - b\sin y_1(t) + l\sin y_1(t)\cos y_1(t) + d\cos\omega t, \end{cases} \tag{2}$$

where $a = \frac{D}{A}$, $b = \frac{rg}{A}$, $d = \frac{F}{A}$ are positive constants, and $l = \frac{3g(B-C)}{R_0 A}$ is a constant.

Let $\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \in \mathbb{R}^2$. Rewrite the system (2) as

$$\dot{\mathbf{y}}(t) = \mathbf{M}\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t)) + \mathbf{m}(t) \tag{3}$$

with

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}, \ \mathbf{m}(t) = \begin{pmatrix} 0 \\ d\cos\omega t \end{pmatrix},$$

$$\mathbf{f}(\mathbf{y}(t)) = \begin{pmatrix} f_1(y(t)) \\ f_2(y(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ -b\sin y_1(t) + l\cos y_1(t)\sin y_1(t) \end{pmatrix}.$$

Let $\varphi(\xi) = \sin\xi - \frac{l}{2b}\sin 2\xi \in \mathbb{R}, \forall \xi \in \mathbb{R}$, and $\mathbf{H} = \begin{pmatrix} 0 \\ -b \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. One can construct a general master-slave synchronization scheme for system (3)

$$\mathcal{M} : \dot{\mathbf{y}}(t) = \mathbf{M}\mathbf{y}(t) + \mathbf{H}\varphi(\mathbf{U}\mathbf{y}(t)) + \mathbf{m}(t) \tag{4}$$

$$\mathcal{S} : \dot{\mathbf{z}}(t) = \mathbf{M}\mathbf{z}(t) + \mathbf{H}\varphi(\mathbf{U}\mathbf{z}(t)) + \mathbf{m}(t) + \mathbf{u}(t) \tag{5}$$

with master system $\mathcal{M}$ and slave system $\mathcal{S}$, where $\mathbf{u}(t)$ is the controller, $\mathbf{z}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \in \mathbb{R}^2$. In [7], Wu *et al.* (2006) used a linear feedback controller $\mathcal{C} : \mathbf{u}(t) = \mathbf{K}(\mathbf{y}(t) - \mathbf{z}(t))$ to study the master-slave synchronization. Due to finite speeds of transmission and spreading, a signal, which travels through the master system, the slave system and a controller, is often associated with a time delay, which means that a time delay occurs during the signal transferring from the master system to the controller as well as during the signal transferring from the slave system to the controller. Observing this fact, we choose the following controller $\mathcal{C}$

$$\mathcal{C} : \mathbf{u}(t) = \mathbf{K}(\mathbf{y}(t-\tau) - \mathbf{z}(t-\tau)) \tag{6}$$

where the time-delay $\tau > 0$. Fig. 2 illustrates the feedback control process for the master-slave synchronization scheme described by (4), (5) and (6).

Defining a signal $\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{z}(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \in \mathbb{R}^2$, where $e_1(t) = y_1(t) - z_1(t), e_2(t) = y_2(t) - z_2(t)$, we have the error system

$$\dot{\mathbf{e}}(t) = \mathbf{M}\mathbf{e}(t) - \mathbf{K}\mathbf{e}(t-\tau) + \mathbf{H}\varphi(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t)) \tag{7}$$

where

$$\begin{aligned} \varphi(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t)) &= \varphi(\mathbf{U}\mathbf{y}(t)) - \varphi(\mathbf{U}\mathbf{z}(t)) \\ &= (\sin y_1(t) - \sin z_1(t)) - \frac{l}{2b}(\sin 2y_1(t) - \sin 2z_1(t)). \end{aligned}$$

From the differential mean value theorem, one can obtain

$$\begin{aligned} & (\sin y_1(t) - \sin z_1(t)) - \frac{l}{2b}(\sin 2y_1(t) - \sin 2z_1(t)) \\ = \; & \cos\xi e_1(t) - \frac{l}{b}\cos\tilde{\xi} e_1(t) \end{aligned} \tag{8}$$

where

$$\xi \in (\min\{y_1(t), z_1(t)\}, \max\{y_1(t), z_1(t)\}),$$

$$\tilde{\xi} \in (\min\{2y_1(t), 2z_1(t)\}, \max\{2y_1(t), 2z_1(t)\}).$$

Let $\underline{L} = -1 - |\frac{l}{b}|, \overline{L} = 1 + |\frac{l}{b}|$. It follows from (8) that $\varphi(\mathbf{Ue}(t), \mathbf{z}(t))$ belongs to a sector $[\underline{L}, \overline{L}]$, i.e., $\forall t \geq 0, \mathbf{e}(t), \mathbf{z}(t)$,

$$(\varphi(\mathbf{Ue}(t), \mathbf{z}(t)) - \underline{L}\mathbf{Ue}(t))^T(\varphi(\mathbf{Ue}(t), \mathbf{z}(t)) - \overline{L}\mathbf{Ue}(t)) \leq 0. \tag{9}$$

The initial condition of system (7) is defined as

$$\mathbf{e}(\theta) = \phi(\theta),\ \theta \in [-\tau, 0],\ \phi \in \mathbb{W} \tag{10}$$

where $\mathbb{W}$ is the Banach space of absolutely continuous functions $[-\tau, 0] \to \mathbb{R}^n$ with square-integrable derivative and with the norm

$$\|\phi\|_W = \left[\|\phi(0)\|^2 + \int_{-\tau}^{0} \|\phi(s)\|^2 ds + \int_{-\tau}^{0} \|\dot{\phi}(\theta)\|^2 d\theta\right]^{1/2}$$

where the vector norm $\|\cdot\|$ represents the Euclidean norm.

Remark 1. It should be pointed out that the time delay occurred in the feedback control process between the master system and the controller may be different from the one between the slave system and the controller, which means that the controller (6) should be modified as

$$\mathcal{C} : \mathbf{u}(t) = \mathbf{K}(\mathbf{y}(t-\tau_1) - \mathbf{z}(t-\tau_2)) \tag{11}$$

where $\tau_1 > 0$ and $\tau_2 > 0$. If $\tau_1 = \tau_2$, synchronization is complete. If $\tau_2 > \tau_1$, it is clear that anticipatory synchronization manifold $\mathbf{z}(t) = \mathbf{y}(t + (\tau_2 - \tau_1))$ is a solution of systems (4), (5) and (11). Let $\breve{\mathbf{e}}(t) = \mathbf{y}(t + (\tau_2 - \tau_1)) - \mathbf{z}(t)$, we have

$$\begin{cases} \dot{\mathbf{y}}(t + (\tau_2 - \tau_1)) = \mathbf{M}\mathbf{y}(t + (\tau_2 - \tau_1)) + \mathbf{H}\varphi(\mathbf{U}\mathbf{y}(t + (\tau_2 - \tau_1)), \\ \dot{\mathbf{z}}(t) = \mathbf{M}\mathbf{z}(t) + \mathbf{H}\varphi(\mathbf{U}\mathbf{z}(t)) + \mathbf{u}(t), \\ \mathbf{u}(t) = \mathbf{K}(\mathbf{y}(t-\tau_1) - \mathbf{z}(t-\tau_2)). \end{cases}$$

The error system is

$$\dot{\breve{\mathbf{e}}}(t) = \mathbf{M}\breve{\mathbf{e}}(t) - \mathbf{K}\breve{\mathbf{e}}(t-\tau_2) + \mathbf{H}\breve{\varphi}(\mathbf{U}\breve{\mathbf{e}}(t)), \tag{12}$$

where $\breve{\varphi}(\mathbf{U}\breve{\mathbf{e}}(t)) = \varphi(\mathbf{U}\mathbf{y}(t + \tau_2 - \tau_1)) - \varphi(\mathbf{U}\mathbf{z}(t))$, and the initial condition is $\breve{\mathbf{e}}(\theta) = \breve{\phi}(\theta), \theta \in [-\max\{\tau_1, \tau_2\}, 0]$. It follows from the sector condition (9) that

$$(\breve{\varphi}(\mathbf{U}\breve{\mathbf{e}}(t) - \underline{L}\mathbf{U}\breve{\mathbf{e}}(t))^T(\breve{\varphi}(\mathbf{U}\breve{\mathbf{e}}(t)) - \overline{L}\mathbf{U}\breve{\mathbf{e}}(t)) \leq 0. \tag{13}$$

If $\tau_2 < \tau_1$, one can see that lag synchronization manifold $\mathbf{z}(t) = \mathbf{y}(t-(\tau_1-\tau_2))$ is a solution of systems (4), (5) and (11). Let $\hat{\mathbf{e}}(t) = \mathbf{y}(t - (\tau_1 - \tau_2)) - \mathbf{z}(t)$, we have

$$\begin{cases} \dot{\mathbf{y}}(t-(\tau_1-\tau_2)) = \mathbf{M}\mathbf{y}(t-(\tau_1-\tau_2)) + \mathbf{H}\varphi(\mathbf{U}\mathbf{y}(t-(\tau_1-\tau_2)), \\ \dot{\mathbf{z}}(t) = \mathbf{M}\mathbf{z}(t) + \mathbf{H}\varphi(\mathbf{U}\mathbf{z}(t)) + u(t), \\ \mathbf{u}(t) = \mathbf{K}(\mathbf{y}(t-\tau_1) - \mathbf{z}(t-\tau_2)). \end{cases}$$

The error system is

$$\dot{\hat{\mathbf{e}}}(t) = \mathbf{M}\hat{\mathbf{e}}(t) - \mathbf{K}\hat{\mathbf{e}}(t-\tau_2) + \mathbf{H}\hat{\varphi}(\mathbf{U}\hat{\mathbf{e}}(t)), \tag{14}$$

where $\hat{\varphi}(\mathbf{U}\hat{\mathbf{e}}(t)) = \varphi(\mathbf{U}\mathbf{y}(t-(\tau_1-\tau_2))) - \varphi(\mathbf{U}\mathbf{z}(t))$, and the initial condition is $\hat{\mathbf{e}}(\theta) = \hat{\phi}(\theta), \theta \in [-\max\{\tau_1, \tau_2\}, 0]$. It follows from the sector condition (9) that

$$(\hat{\varphi}(\mathbf{U}\hat{\mathbf{e}}(t) - \underline{L}\mathbf{U}\hat{\mathbf{e}}(t))^T(\hat{\varphi}(\mathbf{U}\tilde{\mathbf{e}}(t)) - \overline{L}\mathbf{U}\hat{\mathbf{e}}(t)) \leq 0. \tag{15}$$

The method to analyze the stability for (12)-(13) and (14)-(15) is similar to that for (7) and (9). In this paper, we focus on the complete synchronization of horizontal platform systems.

The purpose of this paper is to study the master-slave synchronization for horizontal platform systems and to design the controller (6), i.e., to find the controller gain $\mathbf{K}$, such that the system described by (7), (9) and (10) is globally asymptotically stable, which means that the system described by (4)-(6) synchronizes.

To end this section, we introduce some lemmas that will be used in the proofs of synchronization criteria.

Lemma 1. *(Schur Complement) For matrices $\boldsymbol{Q} = \boldsymbol{Q}^T$, $\boldsymbol{S}$ and $\boldsymbol{R} = \boldsymbol{R}^T$ of appropriate dimensions, the inequality*

$$\begin{pmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{R} \end{pmatrix} > 0$$

holds if and only if

$$\boldsymbol{R} > 0, \quad \boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^T > 0.$$

An $\mathcal{S}$-procedure [13] plays an important role in absolute stability and robust stability theory. There are a number of variations, one of which is used in this paper can be expressed as follows.

Lemma 2. *[14] Let $\boldsymbol{F}_i = \boldsymbol{F}_i^T \in R^{n\times n}, i = 0, 1, 2, \cdots p$. Then the following statement is true*

$$\epsilon^T \boldsymbol{F}_0 \epsilon > 0, \textit{for all } \epsilon \neq 0 \textit{ satisfying } \epsilon^T \boldsymbol{F}_i \epsilon \geq 0$$

if there exist real scalars $\varepsilon_i \geq 0, i = 1, 2, \cdots, p$ such that

$$\boldsymbol{F}_0 - \sum_{i=1}^{p} \epsilon_i \boldsymbol{F}_i > 0.$$

For $p = 1$, these two statements are equivalent.

Lemma 3. *[15] For any constant matrix $\boldsymbol{R} > 0$, $\boldsymbol{R} = \boldsymbol{R}^T \in \mathbb{R}^{n\times n}$, scalar $\tau > 0$, and vector functions $\boldsymbol{e}$ and $\dot{\boldsymbol{e}} : [-\tau, 0] \to \mathbb{R}^n$ such that the following integration is well defined, then*

$$-\int_{t-\tau}^{t} \dot{\boldsymbol{e}}^T(s)(\tau \boldsymbol{R})\dot{\boldsymbol{e}}(s)ds \leq \begin{pmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau) \end{pmatrix}^T \begin{pmatrix} -\boldsymbol{R} & \boldsymbol{R} \\ \boldsymbol{R} & -\boldsymbol{R} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau) \end{pmatrix}.$$

Lemma 4. *[2] For any matrix $\boldsymbol{W} > 0$, $\boldsymbol{W} = \boldsymbol{W}^T \in \mathbb{R}^{n\times n}$, a nonsingular matrix $\boldsymbol{U} \in \mathbb{R}^{n\times n}$ and a scalar $\mu > 0$, then*

$$-\boldsymbol{U}^{-1}\boldsymbol{W}(\boldsymbol{U}^{-1})^T \leq \mu^2 \boldsymbol{W}^{-1} - \mu \boldsymbol{U}^{-1} - \mu(\boldsymbol{U}^{-1})^T.$$

3. Synchronization criteria

In this section, we will derive master-salve synchronization criteria for horizontal platform systems. It follows from the loop transformation [16] that the global asymptotic stability of the system (7) in the sector $[\underline{L}, \overline{L}]$ is equivalent to that of the system described by

$$\dot{\mathbf{e}}(t) = (\mathbf{M} + \underline{L}\mathbf{H}\mathbf{U}\mathbf{e}(t) - \mathbf{K}\mathbf{e}(t-\tau) + \mathbf{H}\tilde{\varphi}(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t)) \tag{16}$$

in the sector $[0, \overline{L} - \underline{L}]$, where $\tilde{\varphi}(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t))$ satisfies

$$\tilde{\varphi}^T(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t))(\tilde{\varphi}(\mathbf{U}e(t), \mathbf{z}(t)) - (\overline{L} - \underline{L})\mathbf{U}\mathbf{e}(t)) \leq 0, \forall t > 0, \forall \mathbf{e}(t), \mathbf{z}(t). \tag{17}$$

We employ a delay decomposition approach [9, 10] to choose the following

Lyapunov-Krasovskii functional

$$\begin{aligned} V(t,\mathbf{e}_t,\dot{\mathbf{e}}_t) &= \mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t) + \sum_{i=1}^{N}\int_{t-ih}^{t-(i-1)h}\mathbf{e}^T(\xi)\mathbf{Q}_i\mathbf{e}(\xi)d\xi \\ &\quad + \sum_{i=1}^{N}\int_{-ih}^{-(i-1)h}\int_{t+\theta}^{t}\dot{\mathbf{e}}^T(\xi)(h\mathbf{R}_i)\dot{\mathbf{e}}(\xi)d\xi d\theta \end{aligned} \tag{18}$$

where $\mathbf{e}_t$ is defined as $\mathbf{e}_t = \mathbf{e}(t+\theta),\ \theta \in [-\tau, 0]$, and $\mathbf{P} \in \mathbb{R}^{2\times 2}, \mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{Q}_i \in \mathbb{R}^{2\times 2}, \mathbf{Q}_i = \mathbf{Q}_i^T > 0$, $\mathbf{R}_i \in \mathbb{R}^{2\times 2}, \mathbf{R}_i = \mathbf{R}_i^T > 0$ $(i = 1, 2, \cdots, N)$; $h = \frac{\tau}{N}$, N is the positive integer of division on the interval $[-\tau, 0]$ and h is the length of each division.

Notice that $V(t,\mathbf{e}_t,\dot{\mathbf{e}}_t)$ is a quadratic Lyapunov-Krasovskii functional depending on derivatives. A sufficient condition for asymptotic stability of the error system described by (16) and (17) is that there exist $\varepsilon_i > 0$ $(i = 1, 2, 3)$ such that

$$\begin{aligned} &\varepsilon_1\|\mathbf{e}(t)\|^2 \le V(t,\mathbf{e}_t,\dot{\mathbf{e}}_t) \le \varepsilon_2\|\mathbf{e}_t\|_W^2, \\ &\dot{V}(t,\mathbf{e}_t,\dot{\mathbf{e}}_t) \le -\varepsilon_3\|\mathbf{e}(t)\|^2. \end{aligned}$$

We now state and establish the following synchronization criterion.

Proposition 1. *For a given scale $\tau > 0$, the error system described by (16) and (17) is globally asymptotically stable if there exist matrices $\boldsymbol{P} = \boldsymbol{P}^T > 0$,*

$\boldsymbol{Q}_i = \boldsymbol{Q}_i^T > 0$, $\boldsymbol{R}_i = \boldsymbol{R}_i^T > 0$ $(i = 1, 2, \cdots, N)$, *and a scale* $\Lambda > 0$ *such that*

$$\begin{pmatrix} \Delta^{(1)} & \Delta^{(2)} \\ * & \Delta^{(3)} \end{pmatrix} < 0 \tag{19}$$

where

$$\Delta^{(1)} = \begin{pmatrix} \Delta_{11} & \boldsymbol{R}_1 & 0 & \cdots & 0 & -\boldsymbol{PK} & \Delta_{1\ N+2} \\ * & \Delta_{22} & \boldsymbol{R}_2 & \cdots & 0 & 0 & 0 \\ * & * & \Delta_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \Delta_{NN} & \boldsymbol{R}_N & 0 \\ * & * & * & \cdots & * & \Delta_{N+1\ N+1} & 0 \\ * & * & * & \cdots & * & * & -2\Lambda \boldsymbol{I}_2 \end{pmatrix}$$

with

$$\Delta_{11} = \boldsymbol{P}(\boldsymbol{M} + \underline{L}\boldsymbol{HU}) + (\boldsymbol{M} + \underline{L}\boldsymbol{HU})^T \boldsymbol{P} + \boldsymbol{Q}_1 - \boldsymbol{R}_1,$$

$$\Delta_{ii} = -\boldsymbol{R}_{i-1} - \boldsymbol{R}_i + \boldsymbol{Q}_i - \boldsymbol{Q}_{i-1} \quad (i = 2, 3, \cdots, N),$$

$$\Delta_{N+1\ N+1} = -\boldsymbol{Q}_N - \boldsymbol{R}_N, \quad \Delta_{1\ N+2} = \boldsymbol{PH} + (\overline{L} - \underline{L})\,\boldsymbol{U}^T \Lambda,$$

and

$$\Delta^{(2)} = \begin{pmatrix} h(\boldsymbol{M} + \underline{L}\boldsymbol{HU})^T \boldsymbol{R}_1 & h(\boldsymbol{M} + \underline{L}\boldsymbol{HU})^T \boldsymbol{R}_2 & \cdots & h(\boldsymbol{M} + \underline{L}\boldsymbol{HU})^T \boldsymbol{R}_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ -h\boldsymbol{K}^T \boldsymbol{R}_1 & -h\boldsymbol{K}^T \boldsymbol{R}_2 & \cdots & -h\boldsymbol{K}^T \boldsymbol{R}_N \\ h\boldsymbol{H}^T \boldsymbol{R}_1 & h\boldsymbol{H}^T \boldsymbol{R}_2 & \cdots & h\boldsymbol{H}^T \boldsymbol{R}_N \end{pmatrix},$$

$$\Delta^{(3)} = -diag(\boldsymbol{R}_1 \quad \boldsymbol{R}_2 \quad \cdots \quad \boldsymbol{R}_N).$$

Proof. Taking the derivative of $V(t, \mathbf{e}_t, \dot{\mathbf{e}}_t)$ with respect to t along the trajectory of (16) yields

$$\begin{aligned}
\dot{V}(t,\mathbf{e}_t,\dot{\mathbf{e}}_t) &= \mathbf{e}^T(t)((\mathbf{M}+\underline{L}\mathbf{HU})^T\mathbf{P}+\mathbf{P}(\mathbf{M}+\underline{L}\mathbf{HU}))\mathbf{e}(t) \\
&\quad + 2\mathbf{e}^T(t)\mathbf{PH}\tilde{\varphi}(\mathbf{Ue}(t),\mathbf{z}(t)) \\
&\quad - 2\mathbf{e}^T(t)\mathbf{PKe}(t-\tau) \\
&\quad + \sum_{i=1}^{N}\mathbf{e}^T(t-(i-1)h)\mathbf{Q}_i\mathbf{e}(t-(i-1)h) \\
&\quad - \sum_{i=1}^{N}\mathbf{e}^T(t-ih)\mathbf{Q}_i\mathbf{e}(t-ih) \\
&\quad + \sum_{i=1}^{N}\dot{\mathbf{e}}^T(t)(h^2\mathbf{R}_i)\dot{\mathbf{e}}(t) \\
&\quad - \sum_{i=1}^{N}\int_{t-ih}^{t-(i-1)h}\dot{\mathbf{e}}^T(s)(h\mathbf{R}_i)\dot{\mathbf{e}}(s)ds. \qquad (20)
\end{aligned}$$

In view of Lemma 3 and the error system (16), we have

$$-\int_{t-ih}^{t-(i-1)h}\dot{\mathbf{e}}^T(s)(h\mathbf{R}_i)\dot{\mathbf{e}}(s)ds \le \rho^T(t)\begin{pmatrix} -\mathbf{R}_i & \mathbf{R}_i \\ \mathbf{R}_i & -\mathbf{R}_i \end{pmatrix}\rho(t) \qquad (21)$$

and

$$\dot{\mathbf{e}}^T(t)(h^2\mathbf{R}_i)\dot{\mathbf{e}}(t) = \varrho^T(t)\digamma^T(h^2\mathbf{R}_i)\digamma\varrho(t), \qquad (22)$$

where

$$\rho^T(t) = (\mathbf{e}^T(t-(i-1)h) \;\; \mathbf{e}^T(t-ih)),$$

$$\digamma = (\mathbf{M} + \underline{L}\mathbf{H}\mathbf{U} \quad 0 \cdots 0 \quad -\mathbf{K} \quad \mathbf{H}),$$

and

$$\varrho^T(t) = (\varrho_1^T(t) \cdots \varrho_2^T(t))$$

with

$$\varrho_1^T(t) = (\mathbf{e}^T(t) \quad \mathbf{e}^T(t-h) \quad \mathbf{e}^T(t-2h)),$$

$$\varrho_2^T(t)) = (\mathbf{e}^T(t-(N-1)h) \quad \mathbf{e}^T(t-Nh) \quad \tilde{\varphi}^T(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t))).$$

It follows from (20)-(22) that

$$\dot{V}(t, \mathbf{e}_t, \dot{\mathbf{e}}_t) \leq \varrho^T(t)\Theta\varrho(t)$$

where

$$\Theta = \begin{pmatrix} \Theta_{11} & \mathbf{R}_1 & 0 & \cdots & 0 & \Theta_{1\ N+1} & \Theta_{1\ N+2} \\ * & \Theta_{22} & \mathbf{R}_2 & \cdots & 0 & 0 & 0 \\ * & * & \Theta_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \Theta_{NN} & \mathbf{R}_N & 0 \\ * & * & * & \cdots & * & \Theta_{N+1\ N+1} & \Theta_{N+1 N+2} \\ * & * & * & \cdots & * & * & \Theta_{N+2\ N+2} \end{pmatrix}$$

with

$$
\begin{aligned}
\Theta_{11} &= (\mathbf{M} + \underline{L}\mathbf{HU})^T\mathbf{P} + \mathbf{P}(\mathbf{M} + \underline{L}\mathbf{HU}) + \mathbf{Q}_1 - \mathbf{R}_1 \\
&\quad + (\mathbf{M} + \underline{L}\mathbf{HU})^T(h^2\sum_{i=1}^{N}\mathbf{R}_i)(\mathbf{M} + \underline{L}\mathbf{HU}), \\
\Theta_{1\ N+1} &= -\mathbf{PK} - (\mathbf{M} + \underline{L}\mathbf{HU})^T(h^2\sum_{i=1}^{N}\mathbf{R}_i)\mathbf{K}, \\
\Theta_{1\ N+2} &= \mathbf{PH} + (\mathbf{M} + \underline{L}\mathbf{HU})^T(h^2\sum_{i=1}^{N}\mathbf{R}_i)\mathbf{H}, \\
\Theta_{ii} &= -\mathbf{R}_{i-1} - \mathbf{R}_i + \mathbf{Q}_i - \mathbf{Q}_{i-1}\ (i = 2, 3, \cdots, N), \\
\Theta_{N+1\ N+1} &= -\mathbf{Q}_N - \mathbf{R}_N + \mathbf{K}^T(h^2\sum_{i=1}^{N}\mathbf{R}_i)\mathbf{K}, \\
\Theta_{N+1\ N+2} &= -\mathbf{K}^T(h^2\sum_{i=1}^{N}\mathbf{R}_i)\mathbf{H}, \\
\Theta_{N+2\ N+2} &= \mathbf{H}^T(h^2\sum_{i=1}^{N}\mathbf{R}_i)\mathbf{H}.
\end{aligned}
$$

A sufficient condition for the global asymptotic stability of (16) is that there exist matrices $\mathbf{P} > 0$ and $\mathbf{Q}_i > 0$, $\mathbf{R}_i > 0$ $(i = 1, 2 \cdots, N)$ such that

$$\dot{V}(t, \mathbf{e}_t, \dot{\mathbf{e}}_t) \leq \varrho^T(t)\Theta\varrho(t) < 0 \tag{23}$$

for all $\varrho(t) \neq 0$. From (17), for $\Lambda > 0$, one can obtain that

$$-2\Lambda\tilde{\varphi}^T(\mathbf{Ue}(t), \mathbf{z}(t))[\tilde{\varphi}(\mathbf{Ue}(t), \mathbf{z}(t)) - (\overline{L} - \underline{L})\mathbf{Ue}(t)] \geq 0. \tag{24}$$

Then utilizing $\mathcal{S}$-Procedure and (24), one can see that (23) is implied by the existence of a scale Λ such that

$$\begin{aligned}&\varrho^T(t)\Theta\varrho(t) - 2\tilde{\varphi}^T(\mathbf{Ue}(t), \mathbf{z}(t))\Lambda\tilde{\varphi}(\mathbf{Ue}(t), \mathbf{z}(t))\\&+ 2\tilde{\varphi}^T(\mathbf{Ue}(t), \mathbf{z}(t))\Lambda(\overline{L} - \underline{L})\mathbf{Ue}(t) < 0,\end{aligned} \tag{25}$$

for all $\varrho(t) \neq 0$. Rewrite inequality (25) as

$$\varrho^T(t)\Delta\varrho(t) < 0, \tag{26}$$

where

$$\Delta = \begin{pmatrix} \Delta_{11} & \mathbf{R}_1 & 0 & \cdots & 0 & \Delta_{1\ N+1} & \Delta_{1\ N+2} \\ * & \Delta_{22} & \mathbf{R}_2 & \cdots & 0 & 0 & 0 \\ * & * & \Delta_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \Delta_{NN} & \mathbf{R}_N & 0 \\ * & * & * & \cdots & * & \Delta_{N+1\ N+1} & \Delta_{N+1\ N+2} \\ * & * & * & \cdots & * & * & \Delta_{N+2\ N+2} \end{pmatrix}$$

with

$$\Delta_{11}=\Theta_{11},\ \ \Delta_{1\ N+1}=\Theta_{1\ N+1},\ \ \Delta_{1N+2}=\Theta_{1\ N+2}+(\overline{L}-\underline{L})\mathbf{U}^T\Lambda,$$
$$\Delta_{ii}=\Theta_{ii}\ \ (i=2,3,\cdots,N),\ \ \ \Delta_{N+1\ N+1}=\Theta_{N+1\ N+1},$$
$$\Delta_{N+1\ N+2}=\Theta_{N+1\ N+2},\ \ \Delta_{N+2\ N+2}=\Theta_{N+2\ N+2}-2\Lambda\mathbf{I}_2.$$

It follows from Schur complement that $\Delta < 0$ if there exist matrices $\mathbf{P} > 0$, $\mathbf{Q}_i > 0$, $\mathbf{R}_i > 0$ $(i = 1, 2, \cdots N)$, and a scale Λ such that (19). This completes the proof. Q.E.D.

If $N = 1$, then we have the following Lyapunov-Krasovskii functional

$$\begin{aligned} V_1(t,\mathbf{e}_t,\dot{\mathbf{e}}_t) &= \mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t)+\int_{t-\tau}^{t}\mathbf{e}^T(\xi)\mathbf{Q}_1\mathbf{e}(\xi)d\xi \\ &\quad +\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\mathbf{e}}^T(\xi)(\tau\mathbf{R}_1)\dot{\mathbf{e}}(\xi)d\xi d\theta. \end{aligned} \tag{27}$$

Then Proposition 1 implies the following result.

Corollary 1. *For a given scale $\tau > 0$, the error system described by (16) and (17) is globally asymptotically stable if there exist matrices $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{Q}_1 = \boldsymbol{Q}_1^T > 0$, $\boldsymbol{R}_1 = \boldsymbol{R}_1^T > 0$, and a scale $\Lambda > 0$ such that*

$$\begin{pmatrix} (1,1) & -\boldsymbol{PK} & \boldsymbol{PH}+(\overline{L}-\underline{L})\boldsymbol{U}^T\Lambda & \tau(\boldsymbol{M}+\underline{L}\boldsymbol{HU})^T\boldsymbol{R}_1 \\ * & -\boldsymbol{Q}_1-\boldsymbol{R}_1 & 0 & -\tau\boldsymbol{K}^T\boldsymbol{R}_1 \\ * & * & -2\Lambda\boldsymbol{I}_2 & \tau\boldsymbol{H}^T\boldsymbol{R}_1 \\ * & * & * & -\boldsymbol{R}_1 \end{pmatrix}$$

with

$$(1,1) = \boldsymbol{P}(\boldsymbol{M} + \underline{L}\boldsymbol{HU}) + (\boldsymbol{M} + \underline{L}\boldsymbol{HU})^T\boldsymbol{P} + \boldsymbol{Q}_1 - \boldsymbol{R}_1.$$

If we do not consider the third term in (18) and set $N = 1$, the Lyapunov-Krasovskii functional (18) becomes

$$V_2(t, \mathbf{e}_t) = \mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t) + \int_{t-\tau}^{t} \mathbf{e}^T(\xi)\mathbf{Q}_1\mathbf{e}(\xi)d\xi. \tag{28}$$

Then we can conclude the following delay-independent synchronization criterion.

Corollary 2. *For a given scale $\tau > 0$, the error system described by (16) and (17) is globally asymptotically stable if there exist matrices $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{Q}_1 = \boldsymbol{Q}_1^T > 0$, and a scale $\Lambda > 0$ such that*

$$\begin{pmatrix} (1,1) & -\boldsymbol{PK} & \boldsymbol{PH} + (\overline{L} - \underline{L})\boldsymbol{U}^T\Lambda \\ * & -\boldsymbol{Q}_1 & 0 \\ * & * & -2\Lambda\boldsymbol{I}_2 \end{pmatrix}$$

with

$$(1,1) = \boldsymbol{P}(\boldsymbol{M} + \underline{L}\boldsymbol{HU}) + (\boldsymbol{M} + \underline{L}\boldsymbol{HU})^T\boldsymbol{P} + \boldsymbol{Q}_1.$$

Remark 2. Proposition 1, Corollary 1 and Corollary 2 are derived by using information of nonlinear term $\varphi(\mathbf{Ue}(t), \mathbf{z}(t))$. Notice that if $\tau = 0$, then the control (6) reduces to

$$\mathcal{C} : \mathbf{u}(t) = \mathbf{K}(\mathbf{y}(t) - \mathbf{z}(t)). \tag{29}$$

We now consider the master-slave synchronization scheme [7] described by (4), (5) and (29). The corresponding error system is

$$\dot{\mathbf{e}}(t) = (\mathbf{M} - \mathbf{K})\mathbf{e}(t) + \mathbf{H}\varphi(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t)) \tag{30}$$

where $\varphi(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t))$ is the same as that in (7). It follows from the loop transformation [16] that the global asymptotic stability of the system (30) in the sector $[\underline{L}, \overline{L}]$ is equivalent to that of the system described by

$$\dot{\mathbf{e}}(t) = (\mathbf{M} + \underline{L}\mathbf{H}\mathbf{U} - \mathbf{K})\mathbf{e}(t) + \mathbf{H}\tilde{\varphi}(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t)) \tag{31}$$

in the sector $[0, \overline{L} - \underline{L}]$ where $\tilde{\varphi}(\mathbf{U}\mathbf{e}(t), \mathbf{z}(t))$ is the same as that in (17). Choosing the quadratic Lyapunov function

$$V_3(t, \mathbf{e}(t)) = \frac{1}{2}\mathbf{e}^T(t)\mathbf{P}\mathbf{e}(t) \tag{32}$$

where $\mathbf{P} \in \mathbb{R}^{2\times 2}$ and $\mathbf{P} = \mathbf{P}^T > 0$, we can derive a stability criterion for system (31), which can be stated as that the error system described by (31) and (17) is globally asymptotically stable if there exist a matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a scale $\Lambda > 0$ such that

$$\begin{pmatrix} \mathbf{P}(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U}-\mathbf{K})+(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U}-\mathbf{K})^T\mathbf{P} & \mathbf{P}\mathbf{H}+(\overline{L}-\underline{L})\mathbf{U}^T\Lambda \\ * & -2\Lambda\mathbf{I}_2 \end{pmatrix} < 0. \tag{33}$$

Notice that the error system (7) can be rewritten as

$$\dot{\mathbf{e}}(t) = (\mathbf{M} + \mathbf{W}(t))\mathbf{e}(t) - \mathbf{K}\mathbf{e}(t - \tau) \tag{34}$$

where

$$\mathbf{W}(t) = \begin{pmatrix} 0 & 0 \\ w(t) & 0 \end{pmatrix}$$

with

$$w(t) = \frac{-b(\sin y_1(t) - \sin z_1(t)) + \frac{l}{2}(\sin 2y_1(t) - \sin 2z_1(t))}{y_1(t) - z_1(t)}$$

from which one can see that $-b - |l| \leq w(t) \leq b + |l|$. Therefore, the system (34) can be modeled as a polytopic system. Let

$$\mathbf{W}_1 = \begin{pmatrix} 0 & 0 \\ -b - |l| & 0 \end{pmatrix}, \mathbf{W}_2 = \begin{pmatrix} 0 & 0 \\ b + |l| & 0 \end{pmatrix}.$$

It is clear that $\mathbf{W}_1$ and $\mathbf{W}_2$ are the vertices of $\mathbf{W}(t)$.

Choosing Lyapunov-Krasovskii functional (18) and using the similar proof to Proposition 1, we can derive the following synchronization criterion.

Proposition 2. *For a given scale $\tau > 0$, the error system described by (34) and (10) is globally asymptotically stable if there exist matrices $\boldsymbol{P} = \boldsymbol{P}^T > 0$, $\boldsymbol{Q}_i = \boldsymbol{Q}_i^T > 0$, $\boldsymbol{R}_i = \boldsymbol{R}_i^T > 0$ $(i = 1, 2, \cdots, N)$ such that*

$$\begin{pmatrix} \Omega_{(1)} & \Omega_{(2)} \\ * & \Omega_{(3)} \end{pmatrix} < 0 \tag{35}$$

where

$$\Omega_{(1)}=\begin{pmatrix}\Omega_{11} & \boldsymbol{R}_1 & 0 & \cdots & 0 & -\boldsymbol{PK}\\ * & \Omega_{22} & \boldsymbol{R}_2 & \cdots & 0 & 0\\ * & * & \Omega_{33} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ * & * & * & \cdots & \Omega_{NN} & \boldsymbol{R}_N\\ * & * & * & \cdots & * & \Omega_{N+1\ N+1}\end{pmatrix}$$

with

$$\Omega_{11} = \boldsymbol{P}(\boldsymbol{M}+\boldsymbol{W}_j) + (\boldsymbol{M}+\boldsymbol{W}_j)^T\boldsymbol{P} + \boldsymbol{Q}_1 - \boldsymbol{R}_1 \quad (j = 1, 2),$$

$$\Omega_{ii} = -\boldsymbol{R}_{i-1} - \boldsymbol{R}_i + \boldsymbol{Q}_i - \boldsymbol{Q}_{i-1} \quad (i = 2, 3, \cdots, N),$$

$$\Omega_{N+1\ N+1} = -\boldsymbol{Q}_N - \boldsymbol{R}_N,$$

and

$$\Omega_{(2)}=\begin{pmatrix}h(\boldsymbol{M}+\boldsymbol{W}_j)^T\boldsymbol{R}_1 & h(\boldsymbol{M}+\boldsymbol{W}_j)^T R_2 & \cdots & h(\boldsymbol{M}+\boldsymbol{W}_j)^T\boldsymbol{R}_N\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\\ -h\boldsymbol{K}^T\boldsymbol{R}_1 & -h\boldsymbol{K}^T\boldsymbol{R}_2 & \cdots & -h\boldsymbol{K}^T\boldsymbol{R}_N\end{pmatrix} (j=1,2),$$

$$\Omega_{(3)} = diag(-\boldsymbol{R}_1 \quad -\boldsymbol{R}_2 \quad \cdots \quad -\boldsymbol{R}_N).$$

Remark 3. Applying Proposition 1 and Proposition 2 to check the stability of the system (7) can be formulated as the LMI feasibility problem, which can be solved by the interior-point algorithm. The algorithm has a polynomial-time complexity. The total number of scalar decision variables of Proposition 1 is $M_1 = \frac{1}{2}[(2N+1)n^2 + (2N+1)n] + 1$, and the total row size of the

LMIs is $L_1 = (4N + 3)n + 1$, the numerical complexity of Proposition 1 is proportional to $L_1 M_1^3$. For Proposition 2, the total number of scalar decision variables is $M_2 = \frac{1}{2}[(2N+1)n^2 + (2N+1)n]$, the total row size of the LMIs is $L_2 = (6N+3)n$, the numerical complexity of Proposition 2 is proportional to $L_2 M_2^3$. If $N \geq 2$, it is easy to see that the numerical complexity of Proposition 1 is "smaller" than that of Proposition 2. One can also see that the larger N, the larger numerical complexity for both Proposition 1 and Proposition 2.

Remark 4. If $\tau = 0$, then the control (6) reduces to the control (29). If we rewrite the error system (7) as

$$\dot{\mathbf{e}}(t) = (\mathbf{M} + \mathbf{W}(t) - \mathbf{K})\mathbf{e}(t), \tag{36}$$

where $\mathbf{W}(t)$ is the same as that in (34), and choose Lyapunov function (32), we can obtain a synchronization criterion which is Theorem 2 in [7].

4. Controller design

In this section, based on the analysis results in Section 3, we are in a position to address the problem of controller design. Applying Proposition 1, we first establish the following result for (16) and (17).

Proposition 3. *For a given scale $\tau > 0$, the error system described by (16) and (17) is globally asymptotically stable if there exist scalars $\mu_i > 0$ ($i =$*

$1, 2, \cdots, N)$, $\widetilde{\Lambda} > 0$, *and matrices* $\widetilde{\boldsymbol{P}} = \widetilde{\boldsymbol{P}}^T > 0$, $\widetilde{\boldsymbol{Q}}_i = \widetilde{\boldsymbol{Q}}_i^T > 0$, $\widetilde{\boldsymbol{R}}_i = \widetilde{\boldsymbol{R}}_i^T > 0$ $(i = 1, 2, \cdots, N)$, *and a matrix* $\boldsymbol{Y}$ *of appropriate dimensions such that*

$$\begin{pmatrix} \widetilde{\Delta}^{(1)} & \widetilde{\Delta}^{(2)} & 0 \\ * & \widetilde{\Delta}^{(3)} & \widetilde{\Delta}^{(4)} \\ * & * & \widetilde{\Delta}^{(5)} \end{pmatrix} < 0 \tag{37}$$

where

$$\widetilde{\Delta}^{(1)} = \begin{pmatrix} \widetilde{\Delta}_{11} & \widetilde{\boldsymbol{R}}_1 & 0 & \cdots & 0 & -\boldsymbol{Y} & \widetilde{\Delta}_{1\ N+2} \\ * & \widetilde{\Delta}_{22} & \widetilde{\boldsymbol{R}}_2 & \cdots & 0 & 0 & 0 \\ * & * & \widetilde{\Delta}_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \widetilde{\Delta}_{NN} & \widetilde{\boldsymbol{R}}_N & 0 \\ * & * & * & \cdots & * & \widetilde{\Delta}_{N+1\ N+1} & 0 \\ * & * & * & \cdots & * & * & -2\widetilde{\Lambda}\boldsymbol{I}_2 \end{pmatrix}$$

with

$$\widetilde{\Delta}_{11} = (\boldsymbol{M} + \underline{L}\boldsymbol{H}\boldsymbol{U})\widetilde{\boldsymbol{P}} + \widetilde{\boldsymbol{P}}(\boldsymbol{M} + \underline{L}\boldsymbol{H}\boldsymbol{U})^T + \widetilde{\boldsymbol{Q}}_1 - \widetilde{\boldsymbol{R}}_1,$$

$$\widetilde{\Delta}_{ii} = -\widetilde{\boldsymbol{R}}_{i-1} - \widetilde{\boldsymbol{R}}_i + \widetilde{\boldsymbol{Q}}_i - \widetilde{\boldsymbol{Q}}_{i-1} \quad (i = 2, 3, \cdots, N),$$

$$\widetilde{\Delta}_{N+1\ N+1} = -\widetilde{\boldsymbol{Q}}_N - \widetilde{\boldsymbol{R}}_N, \quad \widetilde{\Delta}_{1\ N+2} = \boldsymbol{H}\widetilde{\Lambda} + (\overline{L} - \underline{L})\widetilde{\boldsymbol{P}}\boldsymbol{U}^T,$$

and

$$\widetilde{\Delta}^{(2)} = \begin{pmatrix} h\widetilde{\boldsymbol{P}}(\boldsymbol{M} + \underline{L}\boldsymbol{H}\boldsymbol{U})^T & h\widetilde{\boldsymbol{P}}(\boldsymbol{M} + \underline{L}\boldsymbol{H}\boldsymbol{U})^T & \cdots & h\widetilde{\boldsymbol{P}}(\boldsymbol{M} + \underline{L}\boldsymbol{H}\boldsymbol{U})^T \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ -h\boldsymbol{Y}^T & -h\boldsymbol{Y}^T & \cdots & -h\boldsymbol{Y}^T \\ h\widetilde{\Lambda}\boldsymbol{H}^T & h\widetilde{\Lambda}\boldsymbol{H}^T & \cdots & h\widetilde{\Lambda}\boldsymbol{H}^T \end{pmatrix},$$

$$\widetilde{\Delta}^{(3)} = diag(-2\mu_1 \widetilde{\boldsymbol{P}} \quad -2\mu_2 \widetilde{\boldsymbol{P}} \quad \cdots \quad -2\mu_N \widetilde{\boldsymbol{P}}).$$

$$\widetilde{\Delta}^{(4)} = diag(\mu_1 \widetilde{\boldsymbol{R}}_1 \quad \mu_2 \widetilde{\boldsymbol{R}}_2 \quad \cdots \quad \mu_N \widetilde{\boldsymbol{R}}_N).$$

$$\widetilde{\Delta}^{(5)} = diag(-\widetilde{\boldsymbol{R}}_1 \quad -\widetilde{\boldsymbol{R}}_2 \quad \cdots \quad -\widetilde{\boldsymbol{R}}_N).$$

Moreover, the feedback controller gain matrix is given by $\boldsymbol{K} = \boldsymbol{Y}\widetilde{\boldsymbol{P}}^{-1}$.

Proof. Pre- and post-multiplying both sides of (19) with

$$diag(\underbrace{\mathbf{P}^{-1}, \quad \mathbf{P}^{-1}, \quad \cdots \quad \mathbf{P}^{-1}}_{N+1}, \quad \Lambda^{-1}\mathbf{I}_2, \quad \mathbf{R}_1^{-1}, \quad \mathbf{R}_2^{-1}, \quad \cdots \quad \mathbf{R}_N^{-1}),$$

yield

$$\begin{pmatrix} \hat{\Delta}^{(1)} & \hat{\Delta}^{(2)} \\ * & \hat{\Delta}^{(3)} \end{pmatrix} < 0 \tag{38}$$

where

$$\hat{\Delta}^{(1)} = \begin{pmatrix} \hat{\Delta}_{11} & \mathbf{P}^{-1}\mathbf{R}_1\mathbf{P}^{-1} & 0 & \cdots & 0 & -\mathbf{K}\mathbf{P}^{-1} & \hat{\Delta}_{1\ N+2} \\ * & \hat{\Delta}_{22} & \mathbf{P}^{-1}\mathbf{R}_2\mathbf{P}^{-1} & \cdots & 0 & 0 & 0 \\ * & * & \hat{\Delta}_{33} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & \hat{\Delta}_{NN} & \mathbf{P}^{-1}\mathbf{R}_N\mathbf{P}^{-1} & 0 \\ * & * & * & \cdots & * & \hat{\Delta}_{N+1\ N+1} & 0 \\ * & * & * & \cdots & * & * & -2\Lambda^{-1}\mathbf{I}_2 \end{pmatrix}$$

with

$$\hat{\Delta}_{11} = (\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})\mathbf{P}^{-1}+\mathbf{P}^{-1}(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})^T+\mathbf{P}^{-1}\mathbf{Q}_1\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{R}_1\mathbf{P}^{-1},$$

$$\hat{\Delta}_{ii} = -\mathbf{P}^{-1}(\mathbf{R}_{i-1}+\mathbf{R}_i-\mathbf{Q}_i+\mathbf{Q}_{i-1})\mathbf{P}^{-1} \quad (i = 2, 3, \cdots, N),$$

$$\hat{\Delta}_{N+1\ N+1} = -\mathbf{P}^{-1}\mathbf{Q}_N\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{R}_N\mathbf{P}^{-1},$$

$$\hat{\Delta}_{1\ N+2} = \mathbf{H}\Lambda^{-1} + \mathbf{P}^{-1}(\overline{L} - \underline{L})\mathbf{U}^T,$$

and

$$\hat{\Delta}^{(2)} = \begin{pmatrix} hP^{-1}(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})^T & h\mathbf{P}^{-1}(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})^T & \cdots & h\mathbf{P}^{-1}(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})^T \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ -h\mathbf{P}^{-1}\mathbf{K}^T & -h\mathbf{P}^{-1}\mathbf{K}^T & \cdots & -h\mathbf{P}^{-1}\mathbf{K}^T \\ h\Lambda^{-1}\mathbf{H}^T & h\Lambda^{-1}\mathbf{H}^T & \cdots & h\Lambda^{-1}\mathbf{H}^T \end{pmatrix},$$

$$\hat{\Delta}^{(3)} = -\mathrm{diag}(\mathbf{R}_1^{-1} \quad \mathbf{R}_2^{-1} \quad \cdots \quad \mathbf{R}_N^{-1}).$$

Then applying Lemma 4, and setting $\widetilde{\mathbf{P}} = \mathbf{P}^{-1}$, $\widetilde{\mathbf{Q}}_i = \mathbf{P}^{-1}\mathbf{Q}_i\mathbf{P}^{-1}, \widetilde{\mathbf{R}}_i = \mathbf{P}^{-1}\mathbf{R}_i\mathbf{P}^{-1}, i = 1, 2, \cdots, N$, $\widetilde{\Lambda} = \Lambda^{-1}$, yield (37). This completes the proof. Q.E.D.

Remark 5. In Remark 2 we give a synchronization criterion for horizontal platform systems with state feedback control (29). Pre- and post-multiplying both sides of (33) with $diag(\mathbf{P}^{-1},\ \Lambda^{-1}\mathbf{I}_2)$ and setting $\widetilde{\mathbf{P}} = \mathbf{P}^{-1}, \widetilde{\Lambda} = \Lambda^{-1}$ yield

$$\begin{pmatrix} (\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})\widetilde{\mathbf{P}}+\widetilde{\mathbf{P}}(\mathbf{M}+\underline{L}\mathbf{H}\mathbf{U})^T-\mathbf{Y}-\mathbf{Y}^T & \mathbf{H}\widetilde{\Lambda}+\widetilde{\mathbf{P}}(\overline{L}-\underline{L})\mathbf{U}^T \\ * & -2\tilde{\Lambda}\mathbf{I}_2 \end{pmatrix} < 0. \tag{39}$$

By the LMI (39), we can obtain the controller (29) for (4) and (5). It can be stated as that the error system described by (31) and (17) is globally asymptotically stable if there exist a matrix $\widetilde{\mathbf{P}} = \widetilde{\mathbf{P}}^T > 0$, a matrix $\mathbf{Y}$ of appropriate dimensions, and a scale $\tilde{\Lambda} > 0$ such that (39). Moreover, the feedback controller gain matrix is given by $\mathbf{K} = \mathbf{Y}\widetilde{\mathbf{P}}^{-1}$.

As pointed out in the Section 3, the error system (7) can be rewritten as (34) which is modeled as a polytopic system. We are in a position to design the controller for system (34). Similar to the proof of Proposition 3, one can derive the following result.

Proposition 4. *For a given scale $\tau > 0$, the error system described by (34) and (10) is globally asymptotically stable if there exist scales $\check{\mu}_i > 0$ $(i = 1, 2, \cdots, N)$, and matrices $\check{\boldsymbol{P}} = \check{\boldsymbol{P}}^T > 0$, $\check{\boldsymbol{Q}}_i = (\check{\boldsymbol{Q}}_i)^T > 0$, $\check{\boldsymbol{R}}_i = \check{\boldsymbol{R}}_i^T > 0$ $(i = 1, 2, \cdots, N)$, and a matrix $\check{\boldsymbol{Y}}$ of appropriate dimensions such that*

$$\begin{pmatrix} \check{\Omega}_{(1)} & \check{\Omega}_{(2)} & 0 \\ * & \check{\Omega}_{(3)} & \check{\Omega}_{(4)} \\ * & * & \check{\Omega}_{(5)} \end{pmatrix} < 0, \tag{40}$$

where

$$\check{\Omega}_{(1)} = \begin{pmatrix} \check{\Omega}_{11} & \check{\boldsymbol{R}}_1 & 0 & \cdots & 0 & -\check{\boldsymbol{Y}} \\ * & \check{\Omega}_{22} & \check{\boldsymbol{R}}_2 & \cdots & 0 & 0 \\ * & * & \check{\Omega}_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \check{\Omega}_{NN} & \check{\boldsymbol{R}}_N \\ * & * & * & \cdots & * & \check{\Omega}_{N+1\ N+1} \end{pmatrix}$$

with

$$\check{\Omega}_{11} = (\boldsymbol{M} + \boldsymbol{W}_j)\check{\boldsymbol{P}} + \check{\boldsymbol{P}}(\boldsymbol{M} + \boldsymbol{W}_j)^T + \check{\boldsymbol{Q}}_1 - \check{\boldsymbol{R}}_1 \quad (j = 1, 2),$$

$$\check{\Omega}_{ii} = -\check{\boldsymbol{R}}_{i-1} - \check{\boldsymbol{R}}_i + \check{\boldsymbol{Q}}_i - \check{\boldsymbol{Q}}_{i-1} \quad (i = 2, 3, \cdots, N),$$

$$\check{\Omega}_{N+1\ N+1} = -\check{\boldsymbol{Q}}_N - \check{\boldsymbol{R}}_N,$$

and

$$\check{\Omega}_{(2)} = \begin{pmatrix} h\check{\boldsymbol{P}}(\boldsymbol{M}+\boldsymbol{W}_j)^T & h\check{\boldsymbol{P}}(\boldsymbol{M}+\boldsymbol{W}_j)^T & \cdots & h\check{\boldsymbol{P}}(\boldsymbol{M}+\boldsymbol{W}_j)^T \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ -h\check{\boldsymbol{Y}}^T & -h\check{\boldsymbol{Y}}^T & \cdots & -h\check{\boldsymbol{Y}}^T \end{pmatrix},$$

$$\check{\Omega}_{(3)} = diag(-2\check{\mu}_1\check{\boldsymbol{P}} \quad -2\check{\mu}_2\check{\boldsymbol{P}} \quad \cdots \quad -2\check{\mu}_N\check{\boldsymbol{P}}),$$

$$\check{\Omega}_{(4)} = diag(\check{\mu}_1\check{\boldsymbol{R}}_1 \quad \check{\mu}_2\check{\boldsymbol{R}}_2 \quad \cdots \quad \check{\mu}_N\check{\boldsymbol{R}}_N),$$

$$\check{\Omega}_{(5)} = diag(-\check{\boldsymbol{R}}_1 \quad -\check{\boldsymbol{R}}_2 \quad \cdots \quad -\check{\boldsymbol{R}}_N).$$

Moreover, the feedback controller gain matrix is given by $\boldsymbol{K} = \check{\boldsymbol{Y}}\check{\boldsymbol{P}}^{-1}$.

Proof. Pre- and post-multiplying both sides of (35) with

$$diag(\underbrace{\mathbf{P}^{-1}, \quad \mathbf{P}^{-1}, \quad \cdots \quad \mathbf{P}^{-1}}_{N+1}, \quad \mathbf{R}_1^{-1}, \quad \mathbf{R}_2^{-1}, \quad \cdots \quad \mathbf{R}_N^{-1}),$$

then applying Lemma 4, and setting $\check{\mathbf{P}} = \mathbf{P}^{-1}$, $\check{\mathbf{Q}}_i = \mathbf{P}^{-1}\mathbf{Q}_i\mathbf{P}^{-1}, \check{\mathbf{R}}_i = \mathbf{P}^{-1}\mathbf{R}_i\mathbf{P}^{-1}, i = 1, 2, \cdots, N$, yield (40). This completes the proof. Q.E.D.

5. An example

In order to show the effectiveness of the derived results in this paper, we consider the horizontal platform system (2) in which the parameters are choosed as $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, and $\omega = 1.8$ rad ms-1. The initial conditions of the master and slave systems are $x_1 = x_2 = 0.1$ and $y_1 = y_2 = 0.01$, respectively. Fig. 3 is the simulation result for the state variables of (2), from which one can see that the horizontal platform system (2) has a double scroll attractor.

First, we consider the synchronization problem. In order to show the effectiveness of Proposition 1 and Proposition 2, let the controller gain matrix $\mathbf{K} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$. Applying Proposition 1 and Proposition 2 to this case, we obtain the maximum allowed time delay bound $\tau_{\max}$ for different N, which are listed in Table 1. From this table, we can conclude that Proposition 1 can provide better results than Proposition 2 for $N = 1$, $N = 2$, $N = 3$, $N = 4$ and $N = 5$, respectively, which also means that keeping the nonlinearity in the error system can derive a larger delay bound than transforming the nonlinear error system into a linear time-varying system; one can also see that the larger N, the better $\tau_{\max}$. The cost of growth of maximum allowed time delay bound is the increase of the computing complexity. From Remark 2, we can compare the numerical complexity of Proposition 1 and Proposition 2 for different N. The comparison results are listed in Table 2. From this table, we can conclude that the numerical complexity of Proposition 1 is "smaller" than that of Proposition 2 for $N = 2$, $N = 3$, $N = 4$ and $N = 5$, respectively. One can also see that the larger N, the larger numerical complexity for both Proposition 1 and Proposition 2.

Taking the time delay as $\tau = 0.2318$ s, we depict the simulation results for master, slave and error systems in Fig. 4, Fig. 5, Fig. 6. The initial condition of master system (4) is $(y_{1_0}, y_{2_0}) = (0.1, 0.1)$ and the initial condition of slave system (5) is $(z_{1_0}, z_{2_0}) = (0.01, 0.01)$. From Fig. 4, Fig. 5, and Fig. 6, one can clearly see that the system (16) is globally asymptotically stable, i.e., the master-slave synchronization scheme described by (4), (5), and (6) indeed achieves synchronization.

For $\tau = 0.16$ s, $\tau = 0.18$ s, $\tau = 0.20$ s, $\tau = 0.23$ s, $\tau = 0.28$ s, applying Proposition 3 and Proposition 4 with $N = 1$, respectively, one can derive the different control gain $\mathbf{K}$, the results are listed in Table 3. From this table, one can clearly see that for the fixed time delay τ, the feedback gain derived by Proposition 4 is smaller than that derived by Proposition 3 in the sense of Euclidean norm. As τ increases, the gain $\mathbf{K}$ derived by Proposition 3 and Proposition 4 increases in the sense of Euclidean norm, respectively.

Let $\tau = 0.24$, $\mu = 0.7$, using Proposition 3, we have $\widetilde{\Lambda} = 3.4996 \times 10^{-11}$ and

$$\widetilde{\mathbf{P}} = 10^{-9} \times \begin{pmatrix} 0.1777 & -0.1186 \\ -0.1186 & 0.2235 \end{pmatrix},$$

$$\widetilde{\mathbf{Q}} = 10^{-9} \times \begin{pmatrix} 0.1643 & -0.1807 \\ -0.1807 & 0.1661 \end{pmatrix},$$

$$\widetilde{\mathbf{R}} = 10^{-9} \times \begin{pmatrix} 0.4995 & -0.2534 \\ -0.2534 & 0.5976 \end{pmatrix},$$

$$\mathbf{Y} = 10^{-9} \times \begin{pmatrix} 0.4426 & -0.0783 \\ -0.1668 & 0.4595 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} 3.4929 & 1.5027 \\ 0.6698 & 2.4109 \end{pmatrix}.$$

Let $\tau = 0.24$, $\breve{\mu} = 0.7$, using Proposition 4, we have

$$\breve{\mathbf{P}} = 10^{-9} \times \begin{pmatrix} 0.1529 & -0.0340 \\ -0.0340 & 0.0862 \end{pmatrix},$$

$$\check{\mathbf{Q}} = 10^{-9} \times \begin{pmatrix} 0.0147 & -0.4009 \\ -0.4009 & 0.3185 \end{pmatrix},$$

$$\check{\mathbf{R}} = 10^{-9} \times \begin{pmatrix} 0.4796 & -0.0808 \\ -0.0808 & 0.1737 \end{pmatrix},$$

$$\check{\mathbf{Y}} = 10^{-9} \times \begin{pmatrix} 0.3967 & -0.0942 \\ 0.0565 & 0.2037 \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} 2.5776 & -0.0765 \\ 0.9807 & 2.7501 \end{pmatrix}.$$

The simulation results for master, slave and error systems for time delay $\tau = 0.24$ and the feedback controller gain derived by Proposition 3 and the gain derived by Proposition 4 are illustrated in Fig. 7-Fig. 12, respectively, where the initial condition of master system (4) is $(x_{1_0}, x_{2_0}) = (0.1, 0.1)$ and the initial condition of slave system (5) is $(y_{1_0}, y_{2_0}) = (0.01, 0.01)$. Fig. 7-Fig. 12 clearly illustrates that the master and slave systems are synchronized, which means that the design method is effective.

6. Conclusion

We have addressed the problem of master-slave synchronization for horizontal platform systems by using time-delay feedback control. We have employed a delay decomposition approach to derive the synchronization criteria. Based on the synchronization criteria, we have derived some sufficient conditions on the existence of a delayed error feedback controller. Moreover, we have designed the controller by solving a set of LMIs. We have also illustrated the effectiveness of synchronization criteria and the design method through one simulation example.

Acknowledgements

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Figures and tables captions

- Fig. 1. Model of the horizontal platform [4] [5]
- Fig. 2. Synchronization scheme
- Fig. 3. Chaotic attractor of the non-autonomous horizontal platform system (1) with $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, $\omega = 1.8$ rad ms-1.
- Fig. 4. Simulation result for master system with time delay $\tau = 0.2318$ and $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, $\omega = 1.8$ rad ms-1,
- Fig. 5. Simulation result for slave system with time delay $\tau = 0.2318$ and $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, $\omega = 1.8$ rad ms-1,
- Fig. 6. Simulation result for error system with time delay $\tau = 0.2318$ and $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, $\omega = 1.8$ rad ms-1,
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- Fig. 8. Simulation result for slave system with the feedback controller gain derived by Proposition 3, $\tau = 0.24$, and $A = 0.3\ \mathrm{kg\ mm}, B = 0.5\,\mathrm{kg\ mm}, C = 0.2\ \mathrm{kg\ mm}, D = 0.4\,\mathrm{kg\ mms\text{-}1}, r = 0.11559633\,\mathrm{kg\ m}, R_0 = 6378000\ \mathrm{m}, g = 9.8\ \mathrm{ms\text{-}2}, F = 3.4\ \mathrm{N}$, $\omega = 1.8$ rad ms-1.

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- Fig. 10. Simulation result for master system with the feedback controller gain derived by Proposition 4, $\tau = 0.24$, and $A = 0.3\,\mathrm{kg\ mm}, B = 0.5\,\mathrm{kg\ mm}, C = 0.2\ \mathrm{kg\ mm}, D = 0.4\,\mathrm{kg\ mms\text{-}1}, r = 0.11559633\,\mathrm{kg\ m}, R_0 = 6378000\ \mathrm{m}, g = 9.8\ \mathrm{ms\text{-}2}, F = 3.4\ \mathrm{N}$, $\omega = 1.8$ rad ms-1.

- Fig. 11. Simulation result for slave system with the feedback controller gain derived by Proposition 4, $\tau = 0.24$, and $A = 0.3\ \mathrm{kg\ mm}, B = 0.5\,\mathrm{kg\ mm}, C = 0.2\ \mathrm{kg\ mm}, D = 0.4\,\mathrm{kg\ mms\text{-}1}, r = 0.11559633\,\mathrm{kg\ m}, R_0 = 6378000\ \mathrm{m}, g = 9.8\ \mathrm{ms\text{-}2}, F = 3.4\ \mathrm{N}$, $\omega = 1.8$ rad ms-1.

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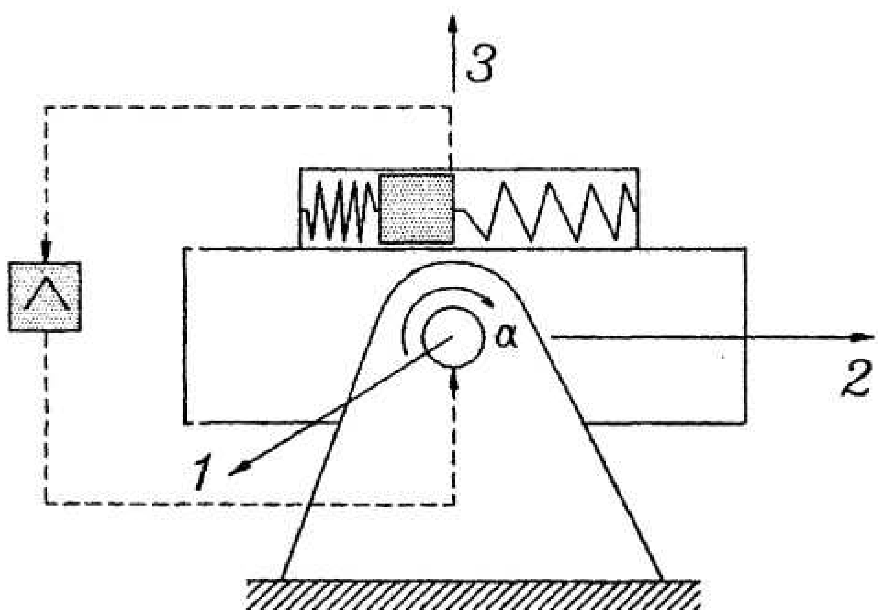
- Table 2. Comparison of the numerical complexity of Proposition 1 and Proposition 2
- Table 3. The feedback controller gain $\mathbf{K}$ for different τ with $N = 1$



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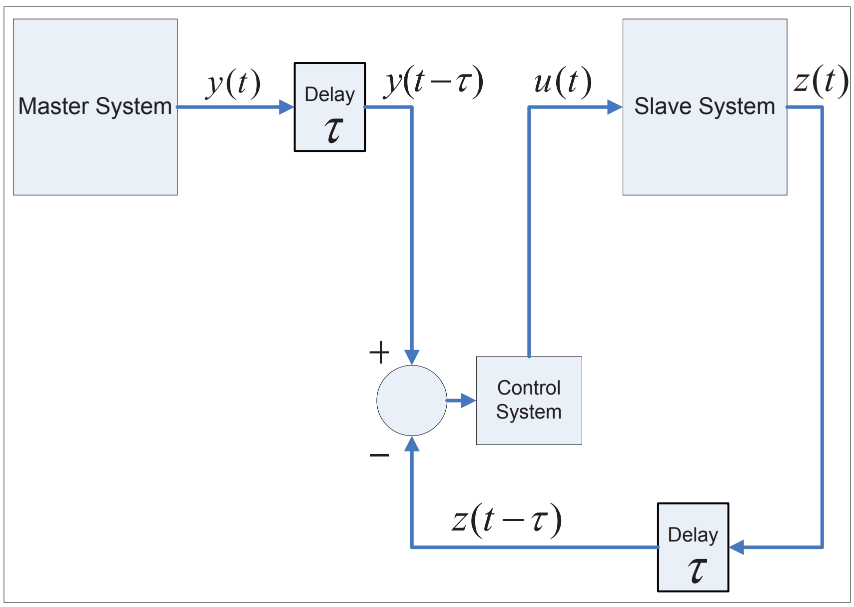
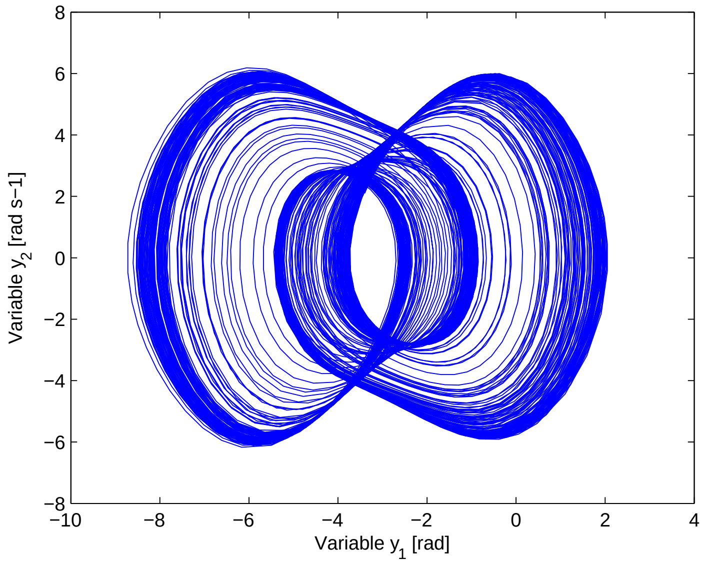


Fig. 2. Synchronization scheme



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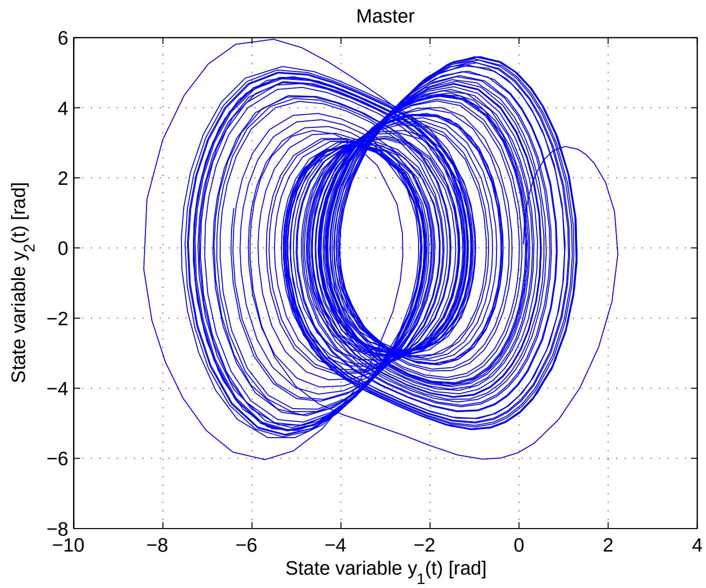

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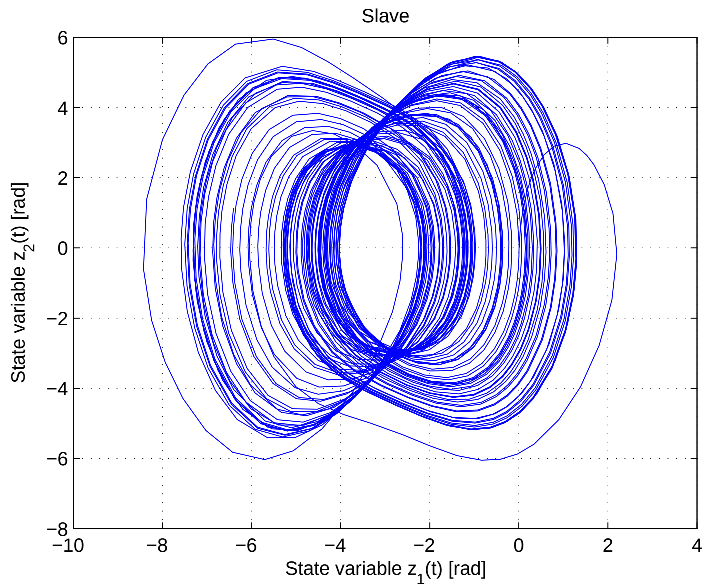

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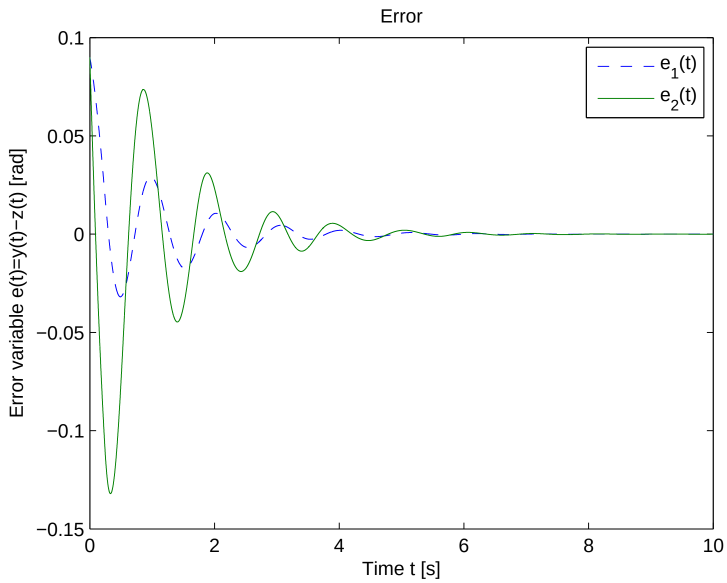
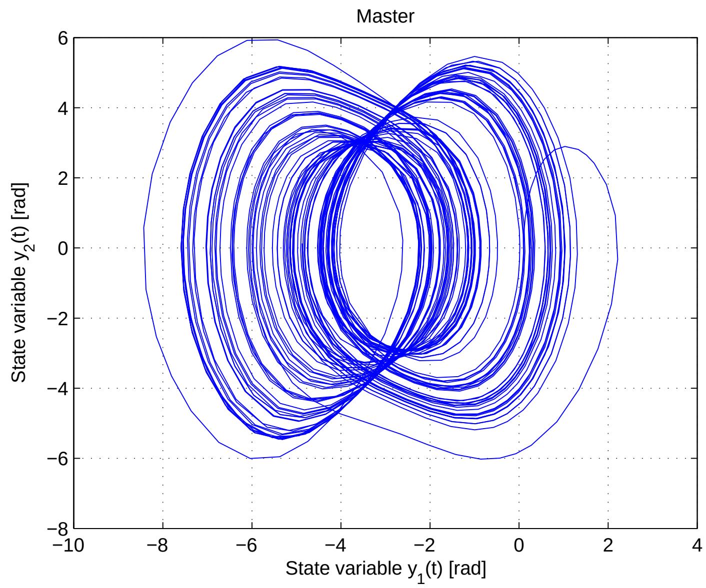


Fig. 6. Simulation result for error system with time delay $\tau = 0.2318$ and $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, $\omega = 1.8$ rad ms-1.



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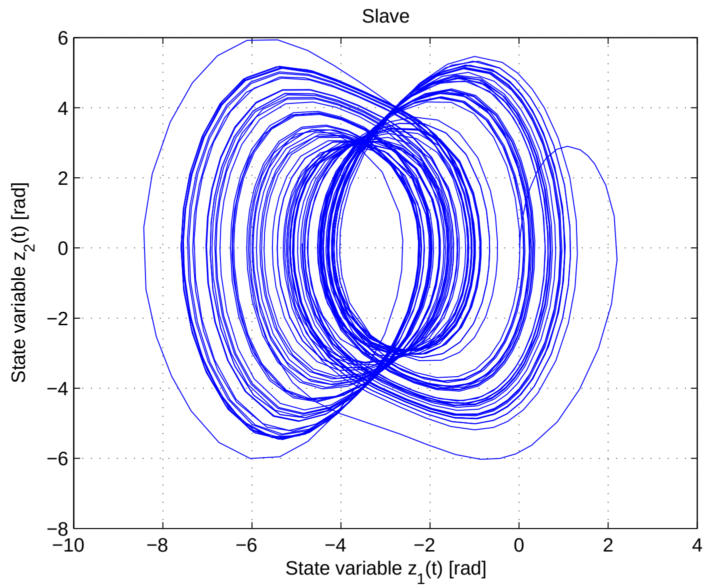
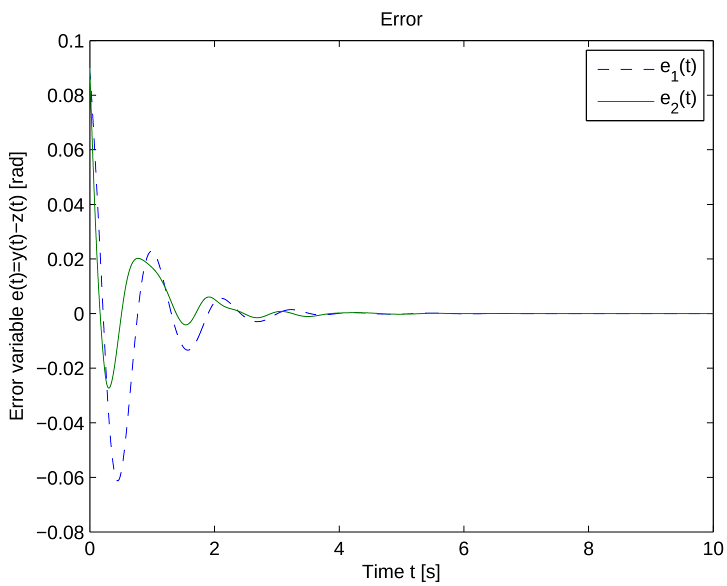


Fig. 8. Simulation result for slave system with the feedback controller gain derived by Proposition 3, $\tau = 0.24$, and $A = 0.3$ kg mm, $B = 0.5$ kg mm, $C = 0.2$ kg mm, $D = 0.4$ kg mms-1, $r = 0.11559633$ kg m, $R_0 = 6378000$ m, $g = 9.8$ ms-2, $F = 3.4$ N, $\omega = 1.8$ rad ms-1.



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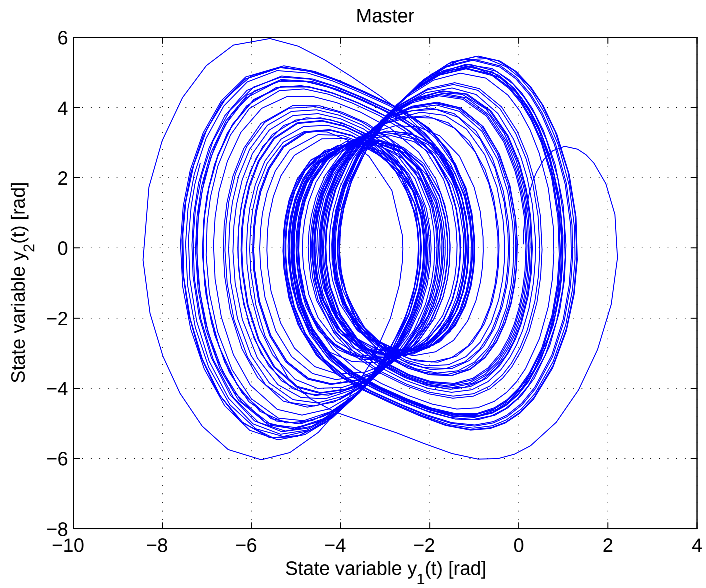

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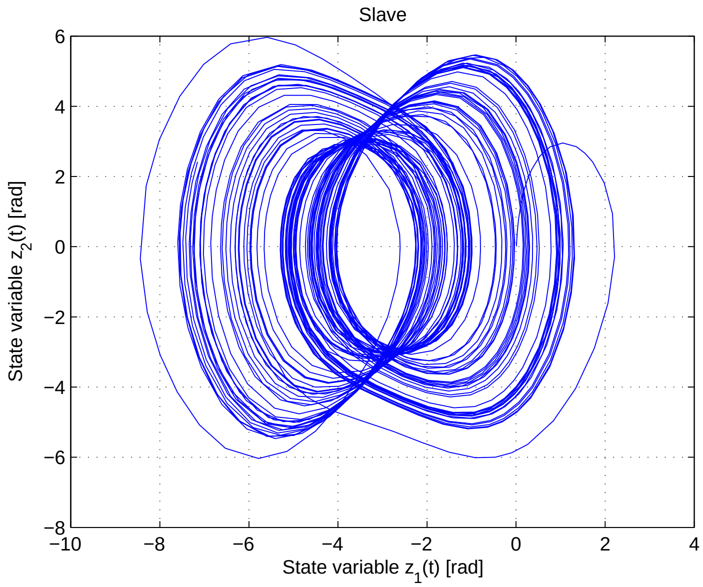

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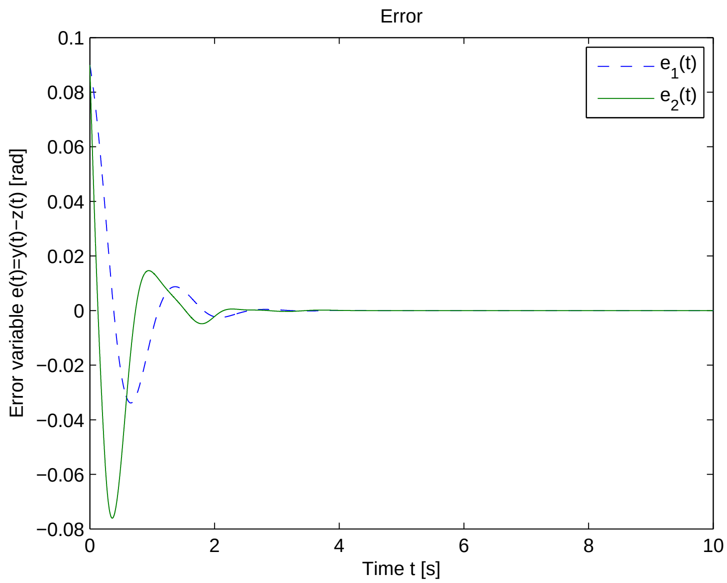

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Table 1
The maximum allowed time delay bound for different N

N	1	2	3	4	5
Proposition 1	0.2166	0.2275	0.2305	0.2313	0.2318
Proposition 2	0.2156	0.2270	0.2298	0.2311	0.2312

Table 2
Comparison of the numerical complexity of Proposition 1 and Proposition 2

		Proposition 1	Proposition 2
N=1	The total number of scalar decision variables	10	9
	The total row size of the LMIs	15	18
	Computing time (s)	1.7813	1.4531
N=2	The total number of scalar decision variables	16	15
	The total row size of the LMIs	23	30
	Computing time (s)	3.3125	3.7969
N=3	The total number of scalar decision variables	22	21
	The total row size of the LMIs	31	42
	Computing time (s)	4.5313	9.1094
N=4	The total number of scalar decision variables	28	27
	The total row size of the LMIs	39	54
	Computing time (s)	6.6250	17.3906
N=5	The total number of scalar decision variables	34	33
	The total row size of the LMIs	47	66
	Computing time (s)	12.5625	26.5938

Table 3

The feedback controller gain $\mathbf{K}$ for different τ with $N = 1$

Time delay	K **derived by Proposition 4**	K **derived by Proposition 4**
$\tau = 0.16$	$\begin{pmatrix} 3.3376 & 1.0821 \\ 0.6333 & 2.7966 \end{pmatrix}$	$\begin{pmatrix} 2.4699 & -0.0640 \\ 0.69176 & 2.9821 \end{pmatrix}$
$\tau = 0.18$	$\begin{pmatrix} 3.3700 & 1.1887 \\ 0.6690 & 2.7164 \end{pmatrix}$	$\begin{pmatrix} 2.5224 & -0.0262 \\ 0.9045 & 2.8820 \end{pmatrix}$
$\tau = 0.20$	$\begin{pmatrix} 3.4383 & 1.3254 \\ 0.6320 & 2.5878 \end{pmatrix}$	$\begin{pmatrix} 2.5995 & 0.0210 \\ 0.8660 & 2.8084 \end{pmatrix}$
$\tau = 0.22$	$\begin{pmatrix} 3.4830 & 1.4392 \\ 0.6229 & 2.4719 \end{pmatrix}$	$\begin{pmatrix} 2.6345 & 0.0226 \\ 0.8706 & 2.7571 \end{pmatrix}$
$\tau = 0.24$	$\begin{pmatrix} 3.4929 & 1.5027 \\ 0.6698 & 2.4109 \end{pmatrix}$	$\begin{pmatrix} 2.5776 & -0.0765 \\ 0.9807 & 2.7501 \end{pmatrix}$