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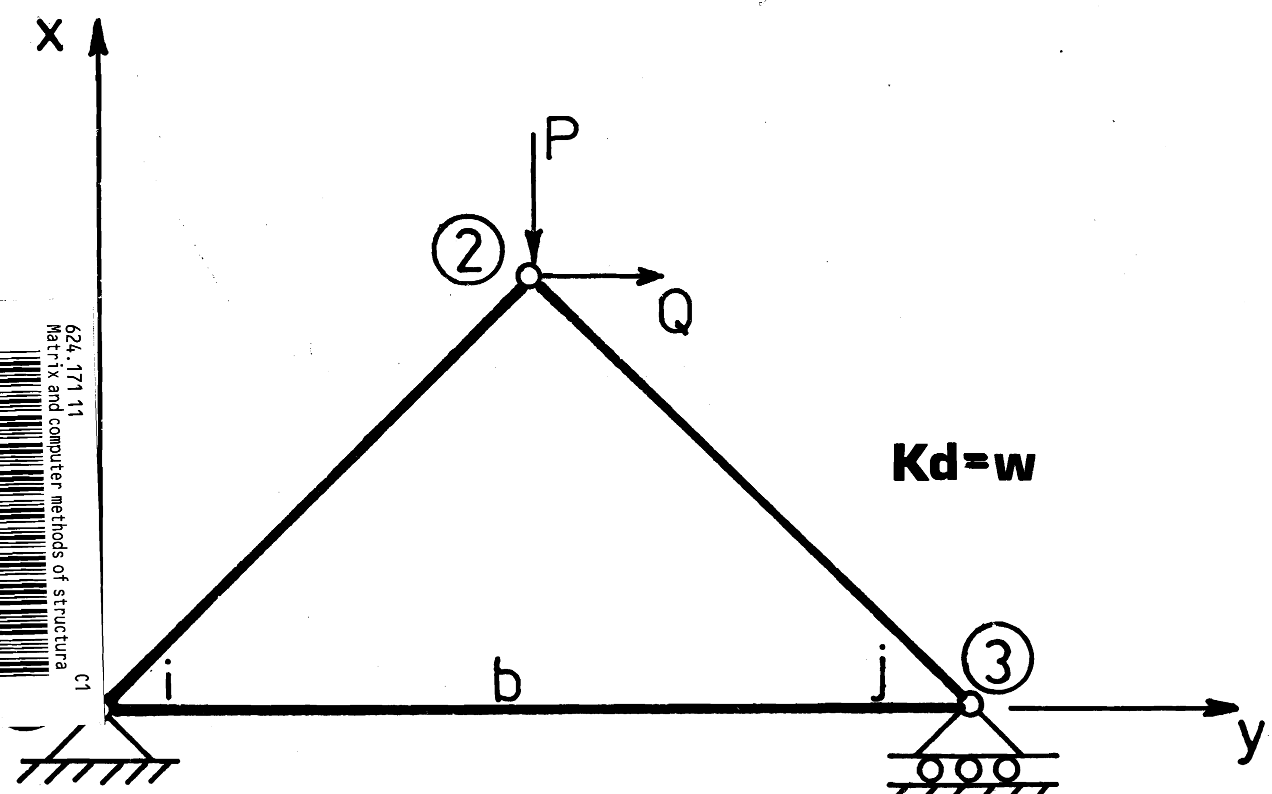
MATRIX & COMPUTER METHODS OF STRUCTURAL ANALYSIS

BOOK 1 — MATRIX ALGEBRA AND THE STIFFNESS METHOD



$$p_{jb} = k21_b d_{ib} + k22_b d_{jb}$$

C.G. McDOWALL

CIVIL ENGINEERING, JULY 1975.

MATRIX AND COMPUTER METHODS

OF

STRUCTURAL ANALYSIS

BOOK 1 - MATRIX ALGEBRA AND THE STIFFNESS METHOD.

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C.G. McDowall

C.I.A.E., July, 1975.

PREFACE

This book is not intended to replace standard texts on the subjects of Matrix Algebra and Matrix Methods of Structural Analysis but rather to supplement them. The book has been prepared in an endeavour to present clearly and concisely the basic concept of the Stiffness Method without cloaking it within a haze of mathematical mystique. Further, it is hoped that by using simple examples and generalisations the full power of the method is realised. The chapter dealing with matrix algebra has been included under one cover primarily for the convenience of the student.

In a quest for simplicity of presentation much of the elegance of the matrix formulation is lost by not using a rigorous matrix approach. However, it is felt in the first instance it is the concepts that are important, and in order to present these clearly some loss of rigor is warranted. Having digested the fundamentals the task of further reading and wider appreciation can then be pursued at leisure.

In this age of the computer revolution it is essential that engineers avail themselves of this "number crunching" phenomena. In terms of Structural Engineering this means that the Designer has been liberated and can now devote his full attention to the basic problem of selection of structural form and no longer be constrained to a "plane frame mentality" attitude which was, unfortunately, the product of having to fumble with tiresome hand solutions.

Finally,it could be said that the Structure Stiffness Matrix is the heart and soul of any structural system and if any so called "feel" is to be developed for structural response it must be attained through the familiarity the Engineer has for the Structure Stiffness Matrix and not his knowledge of individual member response. This matrix is so fundamental to the structure it is envisaged that in the future certain parameters will be fed into a computer program and the optimum Structure Stiffness Matrix will be generated together with a visual display of the associated structural form.

CHAPTER 1.

MATRIX ALGEBRA.

1.1 INTRODUCTION

Matrix algebra has a number of decided advantages over other methods of formulation particularly when applied to problems in Engineering, Physics and the Social Sciences. Its beauty of exposition when portraying the physical problem is certainly one such advantage. This is born out in the clearness and brevity of presentation. Matrix formulation, once fully grasped, also allows a conceptual understanding of important principles to a depth not attainable using supposedly simpler but less spatially descriptive techniques.

1.2 DEFINITION AND EXAMPLES OF MATRICES

Different people, all of whom are aware of what a matrix is in the mathematical context of the word, could well conceive different pictures in their mind of the physical representation of such a matrix. Although the mind picture would be basically the same for each person, the model the matrix represented in each case could be entirely different. The part of the visualisation that would be the same for each person would be that immediately the word matrix was mentioned each would visualise in general "a rectangular array of numbers." The difference in thought patterns would arise in the association of the array with a physical model. For instance, to some people the matrix may represent a table of results whilst some may see it simply as a rectangular array of numbers. Since a matrix is such a general quantity it is best to think of it in the last way.

1.3 DEFINITION

A matrix can be defined as an array of terms such as those shown in Equation 1.1

$$A = \begin{array}{c} \\ m \downarrow \end{array} \overset{\longrightarrow n}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ a_{31} & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{41} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}} \quad \text{----}(1.1)$$

Each of a_{11} through a_{mn} is termed an ELEMENT of the matrix A. An element can represent a pure number, a constant, a variable or a combination of all three. The matrix A of Equation 1.1 contains m rows and n columns. The ORDER OF A MATRIX is defined by the number of rows times the number of columns it contains. In this case A is an m x n matrix.

The double subscript associated with each element of a matrix defines the position of the element in the array. For example, the element a_{ij} is situated in the i th row and j th column of A. The first subscript always defines the row and the second the column with which the element is associated.

The part of A containing elements of the type a_{ii} (where i = j) is called the diagonal of A and a_{ii} is called a diagonal element.

1.4 EXAMPLES OF MATRICES

A specific case of A could be given by Equation 1.2.

$$A = \begin{bmatrix} a & b & \frac{z}{4} & 6.8 \\ x^2 & \int_0^2 y^2 dy & 5 & 0 \\ 0 & 800 & 2 & 0.754 \end{bmatrix} \quad \text{----(1.2)}$$

A in this case is of order (3x4) and the particular element $a_{ij} = a_{24} = 0$. If A reduces to a matrix consisting of one row and one column only i.e. to a_{11} a SCALAR results.

If on the other hand A reduces to a single row or a single column a VECTOR results.

$$A = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{bmatrix} = \text{a column vector of order (5x1)}$$

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 & 10 \end{bmatrix} = \text{a row vector of order (1x5)}$$

Current practice is to disregard the row vector and refer to a vector simply as a column vector.

Re-examination of the row and column vector A shows that the column vector is the row vector turned through 90^o or vice versa. In the jargon of matrix algebra one is said to be the TRANSPONSE (written A^T) of the other. Extending the above idea to the matrix of Equation 1.1 simply means that rows and columns are interchanged and that the transpose of A is given by

$$A^T = \begin{bmatrix} a_{11} & a_{21} & . & . & . & a_{m1} \\ a_{12} & a_{22} & . & . & . & a_{m2} \\ . & . & . & . & . & . \\ . & . & . & . & a_{ij} & . \\ . & . & . & . & . & . \\ a_{1m} & a_{2n} & . & . & . & a_{mn} \end{bmatrix}$$

1.5 TYPES OF MATRICES

There are a number of matrices which appear consistently when applying matrix algebra to problems. The more important of these will now be briefly discussed.

NULL MATRIX.

All the elements of this matrix are zero.

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

SQUARE MATRIX.

When the number of rows, m, is equal to the number of columns, n, of a matrix it is said to be square and of order n.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

It should be noted that only square matrices have determinants.

SYMMETRIC MATRIX.

When a square matrix exists such that $a_{ij} = a_{ji}$ then it is said to be symmetric.

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 2 & 2 & 4 \\ -1 & 2 & 3 & 0 \\ 3 & 4 & 0 & 4 \end{bmatrix}$$

SKEW-SYMMETRIC MATRIX.

A matrix A in which $a_{ij} = a_{ji}$ and $a_{ii} = 0$ is a skew or antisymmetric matrix.

TRIANGULAR MATRIX.

A square matrix for which all the elements above or below the leading diagonal zero is said to be either a lower or an upper triangular matrix respectively.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 6 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Lower Triangular Upper Triangular

DIAGONAL MATRIX.

This is a square matrix in which all the off diagonal elements are zero.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

SCALAR MATRIX.

This is a diagonal matrix for which all of the diagonal elements are the same.

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

UNIT MATRIX.

A scalar matrix in which all of the diagonal elements are unity is called a unit or identity matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.6 MATRIX OPERATIONS

In this section the basic mathematical operations which can be performed in matrix algebra will be briefly discussed. It will be seen that the same operations used in scalar algebra still apply. However their application to matrices is completely general.

EQUALITY OF MATRICES

The concept of equality is the most fundamental relationship of any algebra and it is also probably the most difficult to define.

Two matrices are said to be equal IFF they are of the same order and if every element of one equals the corresponding element of the other.

$$A = \begin{bmatrix} a & b & c & d \\ j & k & 1 & m \end{bmatrix} = \begin{bmatrix} 1 & 5 & x^3 & -10 \\ 0 & \int_0^1 x2dx & 5 & \frac{L}{M} \end{bmatrix} \quad (1.3)$$

Equation 1.3 states that $a = 1$, $b = 5$, $c = x^3$ etc.

The equality of matrices of different order is not defined.

The basic blasphemy of matrix algebra is to equate two matrices of different order.

1.7 ADDITION AND SUBTRACTION

Two matrices of the same order may be added together and their sum is defined as a matrix of the same order every element of which is the sum of the corresponding elements of the original matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 5 & 8 & 9 & 1 \\ 11 & 7 & b & 2 \end{bmatrix} + \begin{bmatrix} -1 & 3 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 5 & 0 \\ 7 & 9 & 9 & 1 \\ 13 & 11 & b+3 & 3 \end{bmatrix}$$

Subtaction is defined in a similar manner.

$$\underset{2 \times 3}{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 3 & -1 \end{bmatrix}} - \underset{2 \times 3}{\begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} -2 & 2 & -2 \\ 2 & 2 & -1 \end{bmatrix}}$$

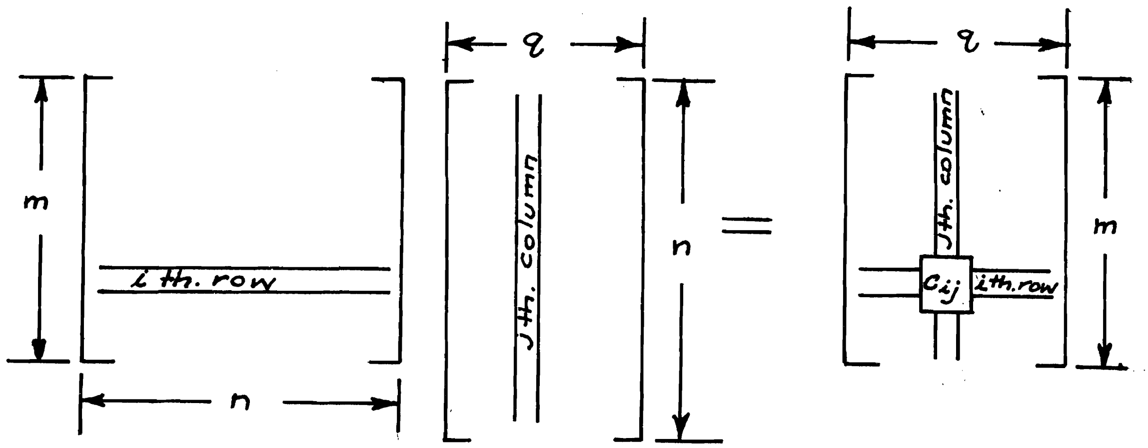
1.8 SCALAR MULTIPLICATION

The product of a scalar and a matrix is defined as a matrix of the same order in which every element is equal to the product of each element and the scalar.

$$2 \times \begin{bmatrix} 5 & 7 & 0 \\ 0 & 2 & -3 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 & 0 \\ 0 & 4 & -6 \\ 6 & 0 & 8 \end{bmatrix}$$

1.9 MATRIX MULTIPLICATION

The product of an (mxn) matrix A and an (nxq) matrix B is defined as an (mxq) matrix C in which a typical element Cij is obtained from the sum of the products of the elements of the ith row of A and the respective elements of the jth column of B.



A X B = C

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 9 & 0 \\ 7 & 12 & 21 & 0 \end{bmatrix}$$

To obtain the element C_{21} requires multiplying the elements of the second row of A by the elements of the first column of B and adding, thus

$$(4x2) + (5x1) + (6x-1) = 8+5-6 = 7.$$

When matrices satisfy the requirements for multiplication they are said to be conformable for multiplication. It should be noted that although matrices may be conformable in one order they may not be taken in the reverse order.

1.10 MATRIX DIFFERENTIATION

Matrix differentiation simply involves the differentiation of each element of the matrix.

$$A = \begin{bmatrix} x & 2x^2 & 5x \\ 6 & \frac{x^3}{3} & 8x \\ 3x & x^2 & 2 \end{bmatrix}$$

$$\frac{d}{dx}\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 1 & 4x & 5 \\ 0 & x^2 & 8 \\ 12x^3 & 2x & 0 \end{bmatrix}$$

1.11 MATRIX INTEGRATION

Integration of matrix is also done by integrating each element of the matrix.

$$\int_0^x \begin{bmatrix} A \end{bmatrix} dx = \begin{bmatrix} \frac{x^2}{2} & 2x^3 & \frac{5}{2}x^2 \\ 6x & \frac{x^4}{12} & 4x^2 \\ \frac{3}{5}x^5 & \frac{x^3}{3} & 2x \end{bmatrix}$$

The need may arise for the integration of more complex expressions such as

$$\int_0^x \int_0^y \begin{bmatrix} B^T & D & B \end{bmatrix} dx \quad dy$$

To perform such an integration requires that the matrix multiplication first be performed then the integration performed as described in section 4.6.

1.12 MATRIX INVERSION

Matrix inversion is analogous to division in scalar algebra. Consider the scalar equation

$$a.\ b = 1$$

$$a = \frac{1}{b} = b^{-1}$$

i.e. the reciprocal of b is equal to a.

Consider now the matrix equation

$$A.\ B = I.$$

Since the above equality does exist although it is not permissable to write

$$A = \frac{I}{B}$$

it is acceptable to write $A = B^{-1}$

Hence although division and inversion are analogous they are certainly not synonymous.

It should be noted that only square matrices possess an inverse and that the inverse is itself square. Also only non-singular matrices can be inverted i.e. matrices whose determinant is non-zero.

By definition the inverse of a matrix is obtained by the application of equation

$$\boxed{A^{-1} = \frac{1}{\text{DET. } A} \text{ ADJ. } A} \quad \text{---- (1.4)}$$

EXAMPLE 1. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

SOLUTION

$$A = 1\begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} +3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 1(3x3 - 4x4) \;\; -2(1x3 - 1x4) \;\; + 3(1x4 - 1x3)$$

$$= -7 + 2 + 3$$

$$A = -2 = \text{Det. } A.$$

To obtain the adjoint of A requires that the cofactors of A be first determined. However, before finding the cofactors it is necessary to define another term used in determinant theory i.e. the minor of a determinant.

Returning to matrix A, its determinant can be written as

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{vmatrix}$$

The minors of A are obtained by suppressing all the elements of the ith row and jth column of the determinant, thus Mij remains. Hence for A, the minors become

$$M_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} \; ; \; M_{12} = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \; ; \; M_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \text{ etc.}$$

To obtain the cofactor of any element a_{ij} requires assigning the correct sign to the associated minor, thus

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

for this example

$$C_{11} = (-1)^2 M_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -7$$

$$C_{12} = (-1)^3 M_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1$$

$$C_{13} = (-1)^4 M_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1 \text{ etc.}$$

the adjoint of A then becomes the <u>transpose</u> of the matrix made up of the cofactors of A hence

$$Cij = \begin{bmatrix} C11 & C12 & C13 \\ C21 & C22 & C23 \\ C31 & C32 & C33 \end{bmatrix}$$

resulting in

$$\text{Adj. } A = \begin{bmatrix} C11 & C21 & C31 \\ C12 & C22 & C32 \\ C13 & C23 & C33 \end{bmatrix}$$

Evaluating each of the cofactors of A results in

$$Cij = \begin{bmatrix} -7 & 1 & 1 \\ 6 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

taking the transpose of Cij gives

$$\text{Adj. } A = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

by definition, the inverse of A is determined by dividing each of the elements of Adj. A by A

$$A^{-1} = \begin{bmatrix} 3.5 & -3 & 0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix}$$

1.13 THEOREMS OF MATRIX ALGEBRA

When learning any new mathematical process one always has to become familiar with the basic rules associated with the successful application of that process. Matrix algebra is no exception to this rule and a knowledge of the following theorems is considered essential if it is hoped to apply matrix algebra to real problems.

THEOREM 1.

Addition of matrices is both commutative and associative.

$$A+B = B + A \quad \text{(commutative)}$$

$$(A+B)+C = A + (B+C) \quad \text{(associative)}$$

THEOREM 2.

Scalar Multiplication of matrices has the following basic properties

$$\text{(I)} \quad (\alpha + \beta)\, A = \alpha\, A + \beta\, A$$

$$\text{(II)} \quad \alpha\, (A + B) = \alpha\, A + \alpha\, B$$

$$\text{(III)} \quad \alpha\, (\beta A) = (\alpha\beta)\, A$$

THEOREM 3.

Matrix multiplication is distributive over addition.

$$A\,(B+C) = AB + AC$$

$$(A+B)\,C + AC + BC$$

THEOREM 4.

Matrix multiplication is associative.

$$ABCD = E$$

$$(AB)\,(CD) = E$$

$$(A(BC)D) = E$$

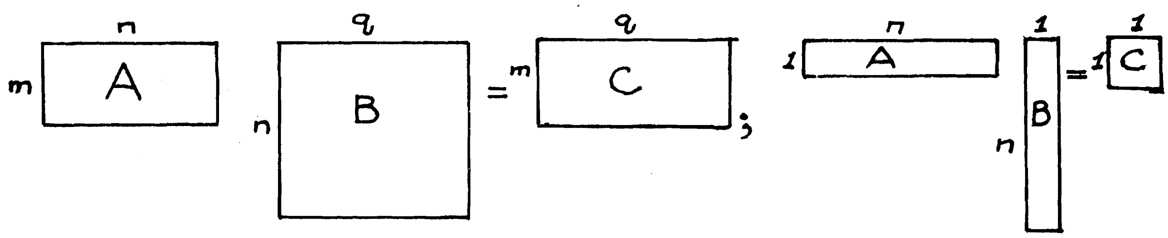
THEOREM 5.

Matrix multiplication is not commutative.

$$A\,B \neq AB$$

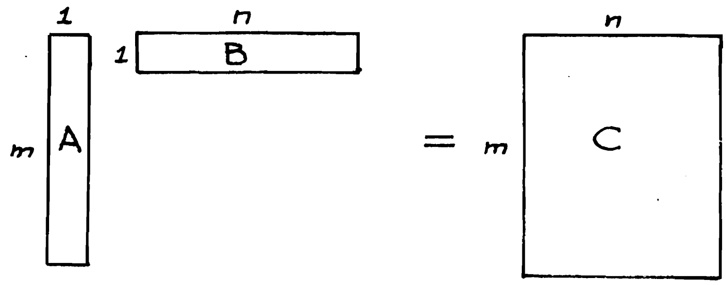
BA

the ways in which the above can occur is shown symbolically below



CASE 1

CASE 2



CASE 3

if $AB = BA$

then it is commutative e.g.

$$A \times A = A \times A$$
$$= A^2$$

also

$AI = A = IA$ (I = A unit matrix of same order as the square matrix A)

and $AO = 0$

$OA = 0$

the above behaviour is once again analogous to that observed in scalar algebra.

THEOREM 6.

The vanishing of the product of two matrices does NOT imply that either of the matrices is a null matrix.

$$AB = 0$$

in the above relationship neither A nor B has to be zero which is quite different from the analogous situation in scalar algebra.

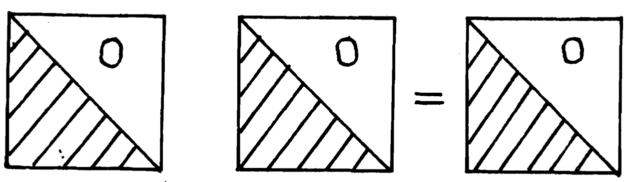
$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ 6 & 3 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

THEOREM 7.

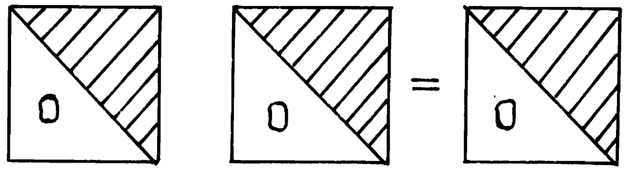
$AB = AC$ does NOT imply that $B = C$.
It can be shown that for A chosen ARBITRARILY then $B = C$.

1.14 PRODUCT OF TRIANGULAR MATRICES

Before continuing with the statement of any further theorems of matrix algebra it is as well to mention the above topic.

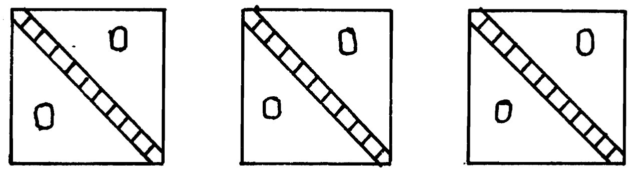


Lower Triangular.



Upper Triangular

In the case of a diagonal matrix



1.15 SOME PROPERTIES OF TRANSPOSED MATRICES

The most significant of these properties can be started thus

(I) $\left[A^T\right]^T = A$

(II) $\alpha A^T = \alpha A^T$

(III) If A is symmetric then

$$A^T = A$$

If $A^T = A$ then A is symmetric.

THEOREM 8.

The transpose of the sum of two matrices is equal to the sum of their transposes

$$(A + B)^T = A^T + B^T$$

THEOREM 9.

The transpose of the product of two matrices is equal to the product of the transposed matrices in the REVERSE ORDER.

$$(AB)^T = B^T A^T$$

This is the FIRST REVERSAL LAW of matrix algebra.

THEOREM 10.

The product of a square non-singular matrix and its inverse taken in either order is equal to a unit matrix.

$$A^{-1} A = AA^{-1} = I.$$

THEOREM 11.

The inverse of a matrix is UNIQUE.

This states that for a square matrix A there exists only one inverse A^{-1}.

THEOREM 11. (Contd.)

PROOF:

assume that there are two inverses A_1^{-1} and A_2^{-1} therefore

$$A_1^{-1} A = I$$

$$A_2^{-1} A = I$$

hence $$A_1^{-1} A = A_2^{-1} A$$

post multiply the system by A_1^{-1}

$$A_1^{-1} A A_1^{-1} = A_2^{-1} A A_1^{-1}$$

but

$$A A_1^{-1} = I$$

thus $$A_1^{-1} I = A_2^{-1} I$$

hence $$A_1^{-1} = A_2^{-1}$$

THEOREM 12.

The inverse of the inverse of a matrix is equal to itself.

$$(A^{-1})^{-1} = A.$$

THEOREM 13.

The inverse of the product of two matrices is equal to the product of the inverses taken in the reverse order.

$$(AB)^{-1} = B^{-1} A^{-1}$$

this is the second reversal law of matrix algebra.

THEOREM 14.

For any non-singular matrix A

$$\left[A^{-1}\right]^{T} = \left[A^{T}\right]^{-1} = A^{-T}$$

THEOREM 15.

The inverse of a symmetric matrix is itself symmetric.

1.16 RANK OF MATRIX

The concept of rank is of fundamental importance in matrix algebra. Two areas in which the rank of a matrix is of particular importance for a complete understanding of the concept involved are as follows:

(1) In the solution of systems simultaneous equations

(2) In explaining determinancy as applied to structural systems.

DEFINITION:

If all the determinants of order greater than r contained in the array of a matrix are zero while at least ONE DETERMINANT of order r is different from zero, the matrix is said to be of rank r.

Example 2.

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$$

SOLUTION.

Since only square matrices have a determinant the rank in this case cannot be greater than 3.

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 4 & -4 \\ -1 & -2 & -1 \end{vmatrix} = 0 \; ; \quad \begin{vmatrix} 1 & -1 & 3 \\ 2 & -4 & 7 \\ -1 & -1 & -2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ -1 & -2 & -2 \end{vmatrix} = 0 \; ; \quad \begin{vmatrix} 2 & -1 & 3 \\ 4 & -4 & 7 \\ -2 & -1 & -2 \end{vmatrix} = 0$$

All of the above determinants are zero. Therefore, the rank of the matrix must be less than 3.

Considering now the determinants of order 2

$$\begin{vmatrix} -1 & 3 \\ -4 & 7 \end{vmatrix} \neq 0$$

$$r_A = 2.$$

NOTE:

The evaluation of the determinants of the given array must be done in an orderly manner.

1.17 LINEAR SIMULTANEOUS EQUATIONS

There are many instances in engineering and science in particular, where the response of the physical system can be represented by a system of simultaneous equations.

1.17 LINEAR SIMULTANEOUS EQUATIONS (Contd.)

Consider as an example of such a system the following set of linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}\,x_2 + a_{13}\,x_3 + \text{-----------} \; a_{1n}\,x_n = b_1 \\ a_{21}x_1 + a_{22}\,x_2 + \text{---------------} \; a_{2n}\,x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}\,x_2 + \text{---------------} \; a_{mn}\,x_n = b_n \end{array}\right\} \text{ --- (1.5)}$$

If all the elements of the R.H.S. of the set is equal to zero then the system is said to be homogeneous. If on the other hand at least ONE element of the R.H.S.is different from zero the system is non-homogeneous.

A set of values x_1, x_2, --- x_n that satisfies the system simultaneously is said to be A SOLUTION to the system.

A solution consisting of all zeros is referred to as the TRIVIAL solution.

A system is said to be CONSISTENT if their exists at least one solution (including the trivial solution) for the system and is said to be INCONSISTENT otherwise.

For a linear system there exists either a UNIQUE solution or INFINITELY many solutions.

Representing the equations of 1.5 in matrix form results in

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or in general

$$\boxed{A \quad x \quad = \quad b} \text{ ----------}$$

A = Coefficient matrix of the system.

x = Solution vector.

b = No particular name in general although in a certain field it may have a name associated with its function.

1.17 LINEAR SIMULTANEOUS EQUATIONS (Contd.)

The matrix formed thus

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$$

is called the AUGMENTED MATRIX.

THE FUNDAMENTAL THEOREM.

A system of m linear simultaneous equations in n unknowns is consistent IF the co-efficient matrix and the augmented matrix have the same RANK r. Furthermore, if r = n then the system has a unique solution and if r is less than n the system has infinitely many solutions.

In order to explain "the fundamental theorem" in physical terms consider first of all the system of simultaneous equations of Equations 1.6.

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 1 \end{aligned}\right\} \text{- - - - - - -} (1.6)$$

In matrix form Equations 1.6 can be written

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

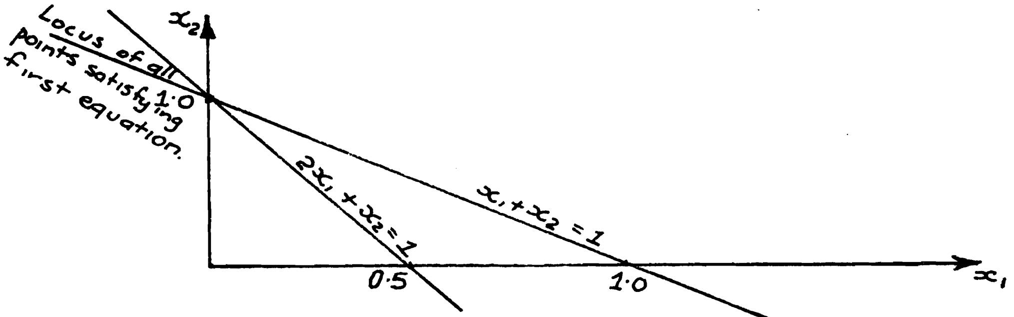
hence

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \text{co-efficient matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \text{augmented matrix}$$

The rank of the co-efficient matrix is 2 which is also the rank of the augmented matrix. Since the number of unknowns is 2 then r = n. Hence a unique solution to the system of equations must exist. This can be illustrated graphically as follows

1.17 LINEAR SIMULTANEOUS EQUATIONS (Contd.)



GEOMETRICAL REPRESENTATION OF EQUATIONS (1.6)

Consider now the system of equations given in Equations 1.7

$$\left.\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned}\right\} \text{- - - - - - - - - -(1.7)}$$

in matrix notation

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

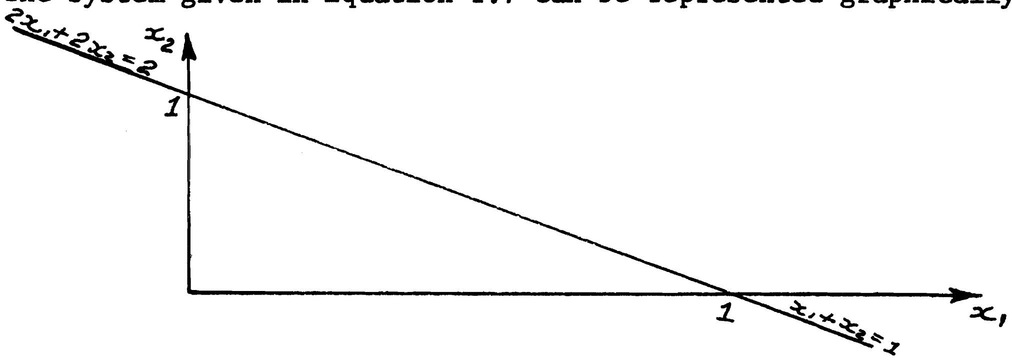
from which

$$r \text{ coeff} = 1, \qquad r \text{ aug} = 1$$

$$r < n$$

$\therefore$ infinitely many solutions exist.

The system given in Equation 1.7 can be represented graphically as follows



GEOMETRICAL REPRESENTATION OF EQUATIONS 1.7

SYSTEMS WITH MORE THAN ONE R.H.S.

There are cases when the R.H.S. of the system of simultaneous equations can consist of more than one vector. Such is the case in Structural Analysis when the structure is subjected to a number of different load cases e.g. dead, live, crane, and wind loads.
For such a case the matrix relationship would appear as

1.17 LINEAR SIMULTANEOUS EQUATIONS (Contd.)

$$\begin{bmatrix} \\ & A & \\ \\ \end{bmatrix} \begin{bmatrix} x_1 & x_1^1 \\ x_2 & x_2^1 \\ | & | \\ | & | \\ x_n & x_n^1 \end{bmatrix} = \begin{bmatrix} b_1 & b_1^1 \\ b_2 & b^1 \\ | & | \\ b_m & b_m^1 \end{bmatrix}$$

or

$$\boxed{A \quad X \quad = \quad B} \text{ - - - - - - - - - - - - (1.8)}$$

Theoretically, at least, the R.H.S. can be as large as you like.

SOLUTION OF SYSTEMS OF SIMULTANEOUS EQUATIONS.

Consider the system of simultaneous equations given by the matrix relationship

$$A \quad x \quad = \quad b \text{ - - - - - - - - - (1.9)}$$

If the system has a unique solution then A is non-singular. Therefore, pre-multiplication of Equation 1.9 by A^{-1} results in

$$A^{-1} \; A\,x = A^{-1} \; b$$

$$x = A^{-1} \; b$$

where A^{-1} is the inverse of A.

In obtaining the solution to a system of simultaneous equations the inverse of the matrix is NEVER determined because of the excessive amount of work involved.

The usual procedure is to resort to a NUMERICAL METHOD to obtain a solution. Three such numerical methods successfully applied to the field of Structural Analysis are Gauss Elimination, Gauss - Seidel Interation and the Cholesky Square-Root method. In this chapter only one method, that is, the Gauss Elimination will be considered.

EXAMPLE 3.

Solve the following system of simultaneous equations using Gauss Elimination.

$$\begin{aligned} 2\,d_{2x} \qquad\qquad\qquad & - d_{3x} = 1 \\ 2\,d_{2y} & + d_{3x} = -1 \\ - d_{2x} + d_{2y} & + 3d_{3x} = 0 \end{aligned}$$

SOLUTION.

Writing the system of equations in matrix form

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{2y} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Step 1.

Set up the augmented matrix

$$\text{First Pivot} \longrightarrow \begin{bmatrix} \boxed{2} & 0 & -1 & 1 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 3 & 0 \end{bmatrix}$$

Step 2.

Divide the first row of the augmented matrix by the pivot i.e. by a_{11} to reduce the pivot element to unity.

$$\begin{bmatrix} 1 & 0 & -0.5 & 0.5 \\ 0 & 2 & 1 & -1 \\ -1 & 1 & 3 & 0 \end{bmatrix}$$

Step 3.

Add scalar multiples of the elements of each row to the elements of the first row to reduce all of the elements in the first column to zero.

$$\begin{bmatrix} 1 & 0 & -0.5 & 0.5 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & 2.5 & 0.5 \end{bmatrix}$$

NOTE:

a_{21} is zero hence no operation is carried out on the second row. a_{31} and all other elements of the third row are multiplied by (+1) and added to the corresponding elements of the first row to form the elements of the third row.

Step 4.

A new pivot a_{22} is now selected and step 2 performed.

$$\begin{bmatrix} 1 & 0 & -0.5 & 0.5 \\ 0 & 1 & 0.5 & -0.5 \\ 0 & 1 & 2.5 & 0.5 \end{bmatrix}$$

Step 5.

Step 3 is now carried out on the matrix

$$\begin{bmatrix} 1 & 0 & -0.5 & 0.5 \\ 0 & 1 & 0.5 & -0.5 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

The steps 2 through 5 are those steps associated with the forward elimination and result in reducing the original coefficient matrix to an upper triangular matrix.

Step 6.

Add a proportion of the last row to the one before last in order to make the last but one element of the last column of the CO-EFFICIENT matrix zero. Proceed to make the last column of the co-efficient matrix zero.

$$\begin{bmatrix} 1 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & -0.75 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

Step 7.

By taking appropriate scalar multiples of the last but one row reduce the elements of the second last column of the co-efficient matrix to zero.

Continue the process until the matrix takes the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Steps 6 and 7 are those steps associated with the backward substitution.

In the example just attempted the solution to the system of simultaneous equations is given by

$$\begin{bmatrix} d_{2x} \\ d_{2y} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0.75 \\ -0.75 \\ 0.5 \end{bmatrix}$$

1.18 EIGENVALUES AND EIGENVECTORS

The eigen-problem is one of tremendous importance in many branches of engineering as well as in a number of other quantitative type disciplines.

The classical eigen-problem of engineering probably occurs in the field of dynamics. In this context it entails a study of frequencies (eigenvalues) and associated modes (eigenvectors). This topic will be pursued further in Chapter 2 of this book.

Consider the relationship

$$A\,x \;=\; y \quad \text{----------} (1.10)$$

in which A is a square matrix and x and y are vectors of the same dimension.

The following question is now posed concerning Equation 10. Is it possible for x to be transformed to a scalar multiple of itself. If this is so then we require a scalar (λ) such that

$$A\,x \;=\; \lambda x$$

The above relationship can be rewritten thus

$$(A - \lambda I)x = 0 \quad \text{--------} (1.11)$$

A non-trivial solution of Equation 1.11 exists IFF the coefficient matrix is singular for a homogeneous system of equations. Hence

$$\boxed{|A - \lambda I| = 0} \quad \text{--------} (1.12)$$

or in expanded form

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \quad \text{----} (1.13)$$

which is the characteristic equation of A and is a polynomial solution in λ.

Expansion of Equation 1.13 results in the following set of roots (eigenvalues) for which the system of simultaneous equations possess non-zero solutions

$$\lambda_1, \quad \lambda_2, \quad \lambda_3 \; \text{-------} \; \lambda_i \; \text{------} \; \lambda_n.$$

Associated with each λ will be a solution of the form

$${}_1X, \quad {}_2X, \quad {}_3X, \; \text{-------} \; {}_iX \; \text{------} \; {}_nX$$

1.18 EIGENVALUES AND EIGENVECTORS (Contd.)

Where 1X, etc. is a vector thus

$$\begin{bmatrix} 1X1 \\ 1X2 \\ 1Xn \end{bmatrix} \quad ; \quad \begin{bmatrix} 2X1 \\ 2X2 \\ 2Xn \end{bmatrix} \quad ; \quad \text{etc.}$$

and are the eigenvectors of the matrix.

SOME PROPERTIES OF EIGENVALUES AND EIGENVECTORS

(a) EIGENVALUE ZERO

This means the matrix is singular. That is, at least one of the equations is linearly dependent and can be removed from the set.

(b) REPEATED ROOTS

In this case one value of λ is repeated n times. For a well behaved system two eigenvalues results in a two dimensional space as a solution, three in a three dimensional space, etc.

(c) SIMPLE EIGENVALUE

A single eigenvalue is repeated only once in the set of eigenvalues of the matrix.

(d) r - FOLD EIGENVALUES

These are eigenvalues which are repeated r times.

(e) DISTINCT EIGENVALUES

Consider the following set of eigenvalues

$$\underbrace{-1.2 \qquad -1.2}_{\text{2 Fold}} \qquad \underbrace{0 \quad 0 \quad 0}_{\text{3 Fold}} \qquad \underbrace{2 \quad 2}_{\text{2 Fold}} \quad \underbrace{3.2 \qquad 4.5}_{\text{Simple}}$$

which result in

$$-1.2 \qquad\qquad 0 \qquad\qquad 2 \quad 3.2 \quad 4.5$$

where repeated roots are used only once.

(f) EIGENVECTORS

The eigenvectors associated with a particular eigenvalue are usually normalised i.e. they are converted into a unit vector.

1.18 EIGENVALUES AND EIGENVECTORS (Contd.)

(f) EIGENVECTORS (Contd.)

Example

Figure 1 shows an element subjected to a state of plane stress. It is required to determine the principal stresses and the associated planes on which they act.

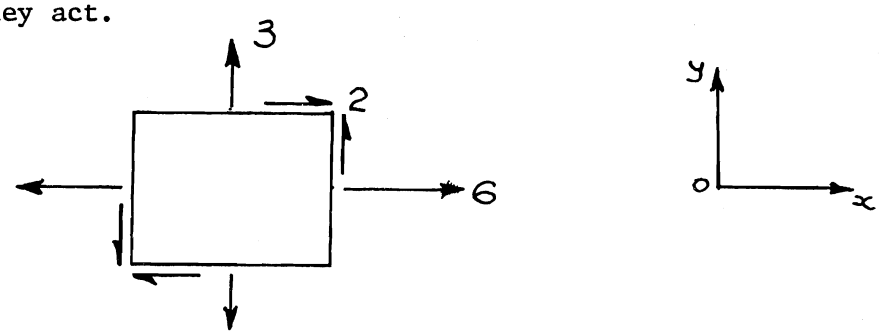


FIGURE 1

The stress tensor for the above state of stress can be written as

$$T_\sigma = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is given by

$$\begin{vmatrix} 6 - \sigma & 2 \\ 2 & 3 - \sigma \end{vmatrix} = 0$$

which on expanding becomes

$$\begin{aligned} (6 - \sigma)(3 - \sigma) - 4 &= 0 \\ 18 - 9\sigma + \sigma^2 - 4 &= 0 \\ \sigma^2 - 9\sigma + 14 &= 0 \end{aligned}$$

The solution of the above quadratic equation gives

$$\sigma_1 = 7 \; ; \quad \sigma_2 = 2$$

which are the eigenvalues.

The associated eigenvectors are determined by substituting each root separately into the characteristic equation thus

$$\begin{bmatrix} 6 - 7 & 2 \\ 2 & 3 - 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$- x_1 + 2x_2 = 0$$

$$x_1 = 2x_2$$

Hence

$${}_1X = \begin{bmatrix} 1 \\ \tfrac{1}{2} \end{bmatrix}$$

1.18 EIGENVALUES AND EIGENVECTORS (Contd.)

(f) EIGENVECTORS (Contd.)

Also

$$\begin{bmatrix} 6 - 2 & 2 \\ 2 & 3 - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$4x_1 + 2x_2 = 0$$

$$x_1 + \tfrac{1}{2}x_2 = 0$$

$$x_1 = \frac{-x_2}{2}$$

$$_2X = \begin{bmatrix} -\tfrac{1}{2} \\ 1 \end{bmatrix}$$

Hence the element is orientated as shown in Figure 2.

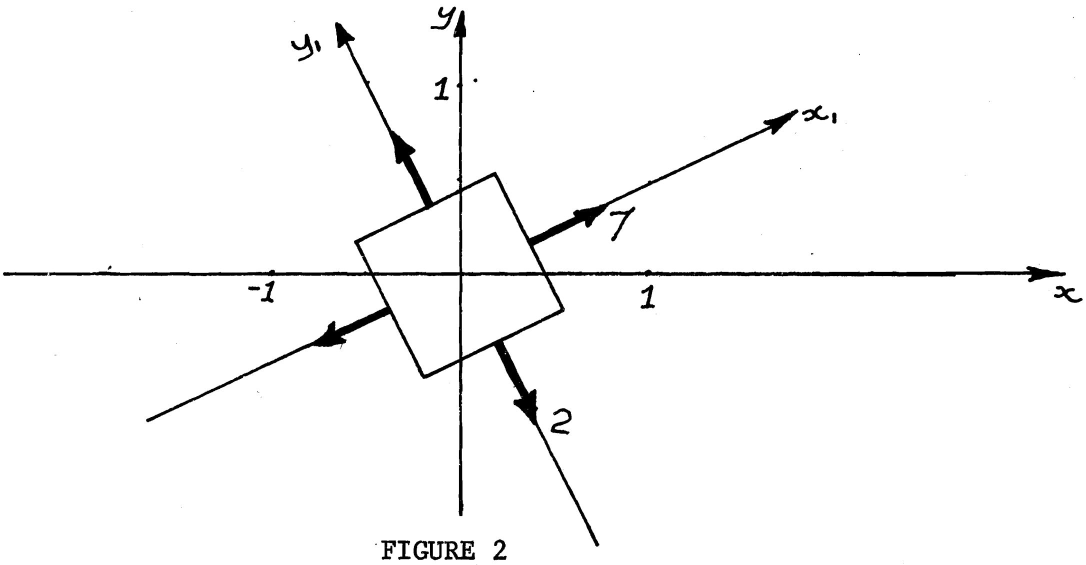


FIGURE 2

From the example it can be seen that the determination of principal stresses is in fact an eigen-problem. Further, it is evident that the magnitude of the principal stresses are the eigenvalues and their direction the eigenvectors. Problems dealing with principal strains and second moments of areas can be solved in a similar manner.

1.19 CONCLUSION

In the foregoing many important aspects of matrix algebra have had to be omitted. However, it is felt that if the student gains a sound understanding of the contents of this chapter at least the burden of further reading on the subject should be considerably lightened.

TUTORIAL PROBLEMS

MATRIX ALGEBRA

QUESTION 1.

Evaluate the following determinant.

$$\begin{vmatrix} 1 & 1 & 2 & 4 \\ 7 & 2 & 3 & 0 \\ 3 & 4 & 1 & 1 \\ 0 & 0 & -2 & 0 \end{vmatrix}$$

QUESTION 2.

Show that the following relationship exists without expanding.

$$\begin{bmatrix} 5 & 4 & 0 \\ 4 & 4 & -3 \\ 11 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 4 & -3 \\ 11 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 4 & 4 & -3 \\ 11 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 & 3 \\ 4 & 4 & -3 \\ 11 & 4 & 0 \end{bmatrix}$$

QUESTION 3.

Given the matrices

$$A = \begin{bmatrix} 5 & 2 & -1 & 3 \\ 2 & 0 & 1 & 8 \\ -7 & 1 & 6 & 9 \end{bmatrix} \; ; \quad B = \begin{bmatrix} 8 & 1 & -1 \\ 0 & 2 & -3 \\ 5 & 0 & 5 \\ 2 & 6 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 7 & -2 \end{bmatrix} \; ; \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{bmatrix}$$

FIND :

$A^T + B$; $B^T - A$; $D^T + D$;

$A \times B$; $B \times A$; $A \times C^T$; $D \times D$;

$D \times A$; $C^T \times C \times B$; $C \times C^T$; $5 \times B$

QUESTION 4.

Show that :

$$(ABC)^T = C^T B^T A^T$$

accepting the first reversal law.

QUESTION 5.

For any matrix A show that A^TA and AA^T are symmetric.

QUESTION 6.

Find the inverses of

$$A = \begin{bmatrix} 4 & 8 & -8 \\ -1 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 4 & 2 & -1 \end{bmatrix} ; \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 2 \\ 3 & 5 & 7 \end{bmatrix}$$

QUESTION 7.

Find the ranks of the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 5 \\ 4 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} ; \quad B = \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 1 & 4 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix} ; \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

QUESTION 8.

Given the following system of linear simultaneous equations

QUESTION 8 (Contd.)

$$\begin{aligned}
-x_1 + 5x_2 + 2x_3 &= 3 \\
x_1 - 6x_2 - x_3 &= 1 \\
2x_1 - 8x_2 + 9x_3 &= -14 \\
x_1 - 3x_2 - 4x_3 &= -b \\
-x_1 + 2x_2 + 5x_3 &= (2b - 7)
\end{aligned}$$

Find the numerical values for (a) and (b) such that

(i) the system is inconsistent

(ii) the system is consistent with infinitely many solutions.

(iii) the system is consistent with a unique solution.

QUESTION 9.

Find the roots of

$$\begin{vmatrix} 1 - \lambda & 2 & -5 \\ 1 & 3 - \lambda & 3 \\ 0 & -2 & 2 - \lambda \end{vmatrix} = 0$$

CHAPTER 2.

THE STIFFNESS METHOD.

2.1 INTRODUCTION

The Stiffness Method is an exact method of Structural Analysis, within the scope of the assumptions made, which uses the generality and elegance of matrix algebra in its formulation. The use of matrix notation has two decided advantages in the treatment of structural problems

(a) brevity of presentation

(b) ease of automating the steps involved.

There are varying ideas as to what constitutes a structure. It could be argued that any material configuration which occupies space and can carry the loads to which it is subjected is a structure. This no doubt would be a very broad definition and would embrace all material objects. For the present we will confine our interests to a very special class of structure which conforms to the following requirements:

(i) it consists of discrete skeletal elements of constant cross-section, i.e. the members are connected at their two ends only.

(ii) all elements consist of a perfect Hookean material.

(iii) all joints act as perfect hinges.

(iv) any external loading is applied as a point load at a joint.

(v) under the influence of the externally applied loads all the active joints of the structure move through small displacements i.e. a linear analysis only will be considered.

2.2 FUNDAMENTALS

In common with any other type of Structural Analysis, the following three considerations must be utilised and constantly kept in mind.

(1) The need for IDENTIFYING the JOINTS of the structure in order to define the topology and geometry of the system to be analysed.

(2) The allocation of a COORDINATE SYSTEM to which we can relate forces, displacements, and reactions.

It should be noted that in general, for structural problems, there are two types of co-ordinate systems

2.2 <u>FUNDAMENTALS</u> (Contd.)

(a) member or local co-ordinates,

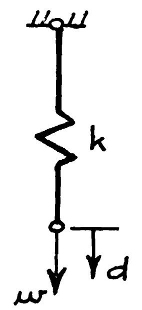
(b) frame or global co-ordinates.

<u>N.B.</u> Initially we will only consider the application of the method in terms of member co-ordinates. This will reduce the conceptual difficulty of the problem but will not cause any real loss of generality.

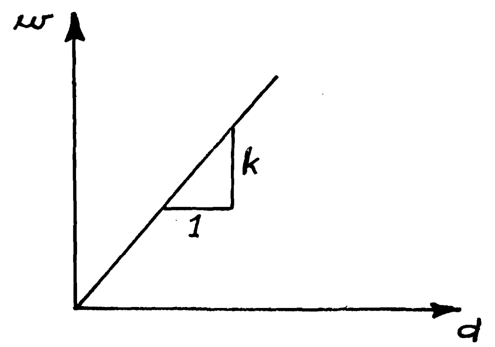
(3) The mutual satisfaction of conditions of <u>EQUILIBRIUM AND COMPATIBILITY</u> at all levels of the structure i.e. for pieces, members, and the structure as a whole.

2.3 <u>STAGE 1</u>

At this point it is desirable to develop the fundamental force - displacement relationship for a linear elastic spring such as is shown in Figure 1(a).



<u>FIGURE 1(a)</u>



<u>FIGURE 1(b)</u>

This can be thought of as the first stage of the formulation process where both equilibrium and compatibility are satisfied at the <u>ELEMENT LEVEL</u>. This is done in fact by developing the constitutive relationships for the material of the elements of the structure.

A plot of w v d results in the response curve shown in Figure 1(b). From this the following well known fundamental stiffness relationship evolves

$$\boxed{kd = w} \quad \text{- - - - - - - - - - - - -} (2.1)$$

The basic form of Equation 2.1 will be continually referred to throughout this chapter.

2.4 <u>APPLICATIONS</u> - <u>SIMPLE LINEAR SPRING SYSTEM</u>

To facilitate a method of attack considered best suited for demonstrating the Stiffness Method, a very simple example will be treated. It should be born in mind that the simplicity of the problem does not greatly detract from its generality of application to skeletal structures.

2.4 APPLICATIONS - SIMPLE LINEAR SPRING SYSTEM (Contd.)

EXAMPLE 1. The simple structure shown in Figure 2 is assumed to consist of a number of Hookean springs pin-connected together at their ends. It is required to determine in general terms the joint displacements, member forces, and reactions.

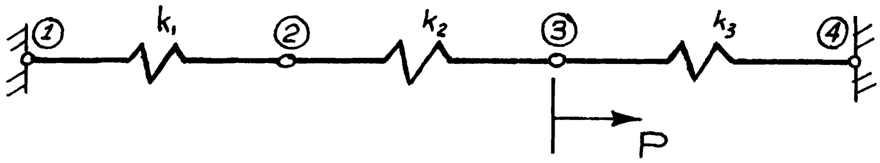


FIGURE 2

SOLUTION: Consider a member (b) isolated from the structure which is assumed to be in equilibrium under the action of the external load system. In general such a member could be represented as shown in Figure 3.

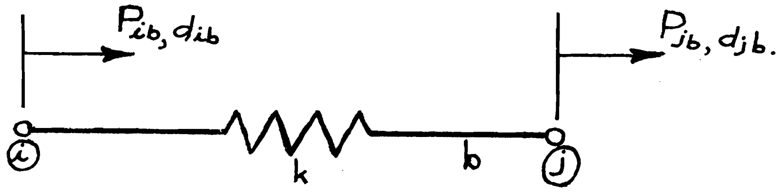


FIGURE 3

2.5 STAGE 2

It is now required that equilibrium and compatibility be satisfied at the MEMBER LEVEL. The most general situation for such a member (b) would be the case where both the (i) and the (j) ends can be displaced. In such a case Equation 2.1 could be written thus

$$\boxed{p = k\Delta} \quad \text{- - - - - - - - - -} \quad (2.2)$$

Where:

p = a vector of the member end forces

k = member stiffness matrix for which as yet the force is unknown

Δ = the change in length of member (b).

From equilibrium considerations of the member shown in Figure 3.

$$p_{ib} + p_{jb} = 0$$

$$\therefore \quad p_{ib} = -p_{jb}$$

from compatibility considerations

2.5 STAGE 2 (Contd.)

$$\Delta = d_{ib} - d_{jb}$$

hence

$$p_{ib} = k_b \Delta_b = k_b d_{ib} - k_b d_{jb}$$

$$p_{jb} = -k_b \Delta_b = -k_b d_{ib} + k_b d_{jb}$$

written in matrix notation

$$\begin{bmatrix} p_{ib} \\ p_{jb} \end{bmatrix} = \begin{bmatrix} k_b & -k_b \\ -k_b & k_b \end{bmatrix} \begin{bmatrix} d_{ib} \\ d_{jb} \end{bmatrix} \qquad \text{-----------(2.3)}$$

or partitioning and generalising

$$\left[\begin{array}{c} p_{ib} \\ \hline p_{jb} \end{array}\right] = \left[\begin{array}{c|c} k11_b & k12_b \\ \hline k21_b & k22_b \end{array}\right] \left[\begin{array}{c} d_{ib} \\ \hline d_{jb} \end{array}\right] \qquad \text{-----------(2.4)}$$

Generalising Equation 2.4 to include any two legged member gives

$$\boxed{P_b = k_b \, d_b} \qquad \text{-----------(2.5)}$$

where:

P_b = a vector of the member end forces whose dimension is dependent on the number of degress of freedom activated.

k_b = a square matrix consisting of the member stiffness sub-matrices k11, k12, etc. Its order is as found for P_b.

d_b = a vector of member end displacements again of the same order as P_b.

POINTS ABOUT EQUATIONS (2.3)

(1) A pin-jointed member of a linearly elastic structural system can be represented in exactly the same way as the spring member of Figure 3 except that in this case

$$k = \frac{EA}{L}$$

Hence Equations 2.4 become, for a pin-jointed member in member coordinates,

$$\left[\begin{array}{c} p_{ib} \\ \hline p_{jb} \end{array}\right] = \left[\begin{array}{c|c} \frac{EA}{L} & \frac{-EA}{L} \\ \hline \frac{-EA}{L} & \frac{EA}{L} \end{array}\right] \left[\begin{array}{c} d_{ib} \\ \hline d_{jb} \end{array}\right]$$

POINTS ABOUT EQUATIONS (Contd.)

(2) The stiffness matrix is symmetric and in general the following relationships exist between the member stiffness sub-matrices for a pin-jointed member

$$k11 = k22$$

$$k21 = k12^T$$

(3) The matrix containing the member stiffness sub-matrices is singular i.e. given a system of forces we cannot obtain a unique solution to the problem. Hence forces and displacements cannot be independent of each other.

(4) Equation (2.3) satisfies both equilibrium and compatibility conditions for the single member.

2.6 STAGE 3

This, the final stage in the generalised formulation, requires the mutual satisfaction of equilibrium and compatibility at the STRUCTURE LEVEL.

Consider now the isolated member 3,4 of the structure shown in Figure 2. The free body diagram of this member and its associated pins is shown in Figure 4.

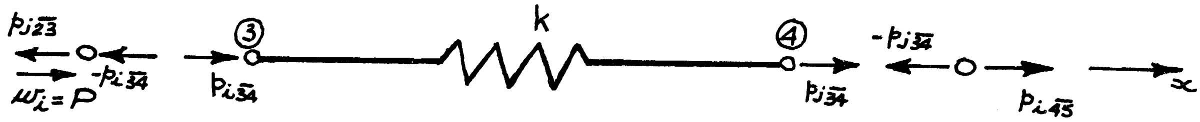


FIGURE 4

From equilibrium consideration of joint 3

$$\Sigma Fx = 0 = -p_{i\overline{34}} - p_{j\overline{23}} + P$$

$$\therefore \; p_{i\overline{34}} + p_{j\overline{23}} = P$$

from Equation (2.4)

$$k11_{\overline{34}}\, d_{i\overline{34}} + k12_{\overline{34}}\, d_{j\overline{34}} + k21_{\overline{23}}\, d_{i\overline{23}} + k22_{\overline{23}}\, d_{j\overline{23}} = P$$

However compatibility conditions of joint (2.3) dictates that

$$d_{j\overline{23}} = d_{i\overline{34}} = d_3$$

$$k11_{\overline{34}}\, d_3 + k12_{\overline{34}}\, d_4 + k21_{\overline{23}}\, d_2 + k22_{\overline{23}}\, d_3 = P \text{ - - - - - (a)}$$

Considering joint 2 and satisfying equilibrium and compatibility

$$p_{j\overline{12}} + p_{i\overline{23}} = 0$$

$$k21_{\overline{12}}\, 3_1 + k22_{\overline{12}}\, d_2 + k11_{\overline{23}}\, d_2 + k12_{\overline{23}}\, d_3 = 0 \text{ - - - - - (b)}$$

from joint 2

$$-p_{i\overline{12}} + r_1 = 0; \quad \therefore \quad p_{i12} = r_1$$

$$k11_{\overline{12}}\, d_1 + k12_{\overline{12}}\, d_2 = r_1 \qquad \text{- - - - -(c)}$$

and finally for joint 2

$$k21_{\overline{34}}\, d_3 + k22_{\overline{34}}\, d_4 = r_4 \qquad \text{- - - - -(d)}$$

Writing equations a, b, c, d in matrix notation results in

$$\begin{array}{c c}
 & \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left[\begin{array}{c|c|c|c}
k11_{\overline{12}} & k12_{\overline{12}} & 0 & 0 \\ \hline
k21_{\overline{12}} & k22_{\overline{12}} + k11_{\overline{23}} & k12_{\overline{23}} & 0 \\ \hline
0 & k21_{\overline{23}} & k22_{\overline{23}} + k11_{\overline{34}} & k12_{\overline{34}} \\ \hline
0 & 0 & k21_{\overline{34}} & k22_{\overline{34}}
\end{array}\right]
\left[\begin{array}{c} d_1 \\ \hline d_2 \\ \hline d_3 \\ \hline d_4 \end{array}\right]
=
\left[\begin{array}{c} r_1 \\ \hline 0 \\ \hline P \\ \hline r_4 \end{array}\right]
\end{array}$$

which can be written

$$\boxed{\bar{K}d = \bar{W}} \qquad \text{- - - - - - - - - - -(2.6)}$$

where K = Primary stiffness matrix of the structure.

d = Displacement vector.

$\bar{W}$ = Appended load vector

POINTS CONCERNING $\bar{K}$

(1) Each row (or column) of $\bar{K}$ satisfies both equilibrium and compatibility at that particular joint of the structure.

(2) $\bar{K}$ is in general a sparse, symmetrix matrix

(3) $\bar{K}$ is singular

(4) The elements of $\bar{K}$ associated with the (i) and (j) joint of the structure represent the stiffness matrix for the member (i, j) of a two-legged member.

$$\begin{array}{c c}
 & \begin{array}{ccccc} & i & & j & \end{array} \\
\begin{array}{c} i \\ \\ j \end{array} & \left[\begin{array}{ccccc} & k11_{ij} & & k12_{ij} & \\ & & & & \\ & k21_{ij} & & k22_{ij} & \end{array}\right]
\end{array}$$

2.7 CONSTRAINT CONDITIONS

To implement the support conditions of a structure is best done by considering a general case and then applying the results to the structure of our example.

Consider the stiffness matrix

$$\left[\begin{array}{c|c|c} \bar{K}_{tt} & \bar{K}_{ts} & \bar{K}_{tu} \\ \hline \bar{K}_{st} & \bar{K}_{ss} & \bar{K}_{su} \\ \hline \bar{K}_{ut} & \bar{K}_{us} & \bar{K}_{uu} \end{array}\right] \left[\begin{array}{c} d_t \\ \hline \delta_s \\ \hline d_u \end{array}\right] = \left[\begin{array}{c} \bar{w}_t \\ \hline r_s \\ \hline \bar{w}_u \end{array}\right]$$

Multiplying out the above matrix relationship results in

$$\begin{array}{ccccccc} \bar{K}_{tt} d_t & + & \boxed{\bar{K}_{ts} \delta_s} & + & \bar{K}_{tu} d_u & = & w_t \\ \bar{K}_{st} d_t & + & \boxed{\bar{K}_{ss} \delta_s} & + & \bar{K}_{su} d_u & = & r_s \\ \bar{K}_{ut} d_t & + & \boxed{\bar{K}_{us} \delta_s} & + & \bar{K}_{uu} d_u & = & w_u \end{array}$$

but for no movement of the supports

$$\delta_s = 0$$

Hence the boxed set of relationships go to zero and in matrix notation we get

$$\begin{bmatrix} \bar{K}_{tt} & \bar{K}_{tu} \\ \bar{K}_{st} & \bar{K}_{su} \\ \bar{K}_{ut} & \bar{K}_{uu} \end{bmatrix} \begin{bmatrix} d_t \\ \delta_s = 0 \\ d_u \end{bmatrix} = \begin{bmatrix} \bar{w}_t \\ r_s \\ \bar{w}_u \end{bmatrix}$$

Therefore there are (n+1) equations in (n) unknowns. Or in other words there is one superfluous equation which can be removed from the set. This then results in the following reduced set of equations

2.7 CONSTRAINT CONDITIONS (Contd.)

$$\left[\begin{array}{c|c} \bar{K}_{tt} & \bar{K}_{tu} \\ \hline \bar{K}_{ut} & \bar{K}_{uu} \end{array}\right] \left[\begin{array}{c} d_t \\ \hline d_u \end{array}\right] = \left[\begin{array}{c} \bar{w}_t \\ \hline \bar{w}_u \end{array}\right]$$

Thus removing the constrained degress of freedom we get

$$\boxed{K\,d = w} \text{ - - - - - - - - - - -(2.7)}$$

where K = Structures stiffness matrix which is non-singular

d = Vector of the active joint displacements

w = External load.

2.8 JOINT DISPLACEMENTS

Returning to our problem, and applying the concepts just discussed to invoke the constraint conditions, it can be seen that this simply requires striking out the row and column associated with the constraint. This is done as follows

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{c|c|c|c} \cancel{k11_{\overline{12}}} & \cancel{k12_{\overline{12}}} & \cancel{0} & \cancel{0} \\ \hline \cancel{k21_{\overline{12}}} & \begin{matrix} k22_{\overline{12}} \\ k11_{\overline{23}} \end{matrix} & k12_{\overline{23}} & \cancel{0} \\ \hline \cancel{0} & k21_{\overline{23}} & \begin{matrix} k22_{\overline{23}} \\ k11_{\overline{34}} \end{matrix} & \cancel{k12_{\overline{34}}} \\ \hline \cancel{0} & \cancel{0} & \cancel{k21_{\overline{34}}} & \cancel{k22_{\overline{34}}} \end{array}\right] \end{array} \left[\begin{array}{c} \cancel{d_1} \\ \hline d_2 \\ \hline d_3 \\ \hline \cancel{d_4} \end{array}\right] = \left[\begin{array}{c} \cancel{r_1} \\ \hline 0 \\ \hline P \\ \hline \cancel{r_4} \end{array}\right]$$

(K), the structure stiffness matrix now becomes

$$\left[\begin{array}{c|c} \begin{matrix} k22_{\overline{12}} \\ + \; k11_{\overline{23}} \end{matrix} & k12_{\overline{23}} \\ \hline k21_{\overline{23}} & \begin{matrix} k22_{\overline{23}} \\ + \; k11_{\overline{34}} \end{matrix} \end{array}\right] \left[\begin{array}{c} d_2 \\ \hline d_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline P \end{array}\right]$$

Solving the above system of simultaneous equations for the displacements results in

2.8 JOINT DISPLACEMENTS (Contd.)

$$\begin{bmatrix} d_2 \\ \\ d_3 \end{bmatrix} = \text{active joint displacements.}$$

We now have a unique set of joint displacements satisfying the conditions of our simple problem.

2.9 MEMBER FORCES

Returning to equations (2.4) and restating them we have

$$\begin{bmatrix} p_{ib} \\ \hline p_{jb} \end{bmatrix} = \left[\begin{array}{c|c} k11_b & k12_b \\ \hline k21_b & k22_b \end{array}\right] \begin{bmatrix} d_{ib} \\ \hline d_{jb} \end{bmatrix}$$

If the second of the above equations is used the member force at the (j) end of the bar can be found. The force found in this way is prefixed by the correct sign. Hence

$$p_{jb} = k21_b d_{ib} + k22_b d_{jb}$$

can now be solved for each of the members $\overline{1,2}$; $\overline{2,3}$; & $\overline{3,4}$. If the solution results in having a negative sign then the member force is compressive, if positive it is tensile. The force at the (i) end is then given by

$$p_{ib} = -p_{jb}$$

2.10 REACTIONS

To find the reactions is simply a matter of applying conditions of local equilibrium to each of the members framing into the reactive joint. For our example

$$R_1 = -p_{i\overline{12}}$$

$$R_4 = -p_{j\overline{34}}$$

2.11 CLOSURE TO EXAMPLE 1

To summarise the procedures adopted in the solution of Example 1 we have the following.

2.11 CLOSURE TO EXAMPLE 1 (Contd.)

(1) Form up the member stiffness sub-matrices of each member of the structure.

(2) Set-up the primary stiffness matrix $\bar{K}$ as a null matrix. The dimensions of $\bar{K}$ for the one degree of freedom joints of our simple structure is (1 x number of joints) x (1 x number of joints).

(3) Place the member stiffness sub-matrices in their correct positions in $\bar{K}$.

(4) Set up the appended load vector $\bar{w}$.

(5) Adjust $\bar{K}$ by implimenting the required constraint conditions.

(6) Step (5) results in $\bar{K}$ being reduced to K to give the relationship

$$Kd = w.$$

(7) Solve the system of simultaneous equations of step (6) for the joint displacements.

(8) Determine the member forces from

$$p_{jb} = k21_a d_{ib} + k22_b d_{jb}$$

and

$$p_{ib} = -p_{jb}$$

(9) Obtain the reactions from local equilibrium considerations.

(10) Statical indeterminacy has not been mentioned.

2.12 APPLICATIONS - SIMPLE TRIANGULATED PIN-JOINTED TRUSS

To generalise the approach formulated in Example 1 and thus enable pin-jointed structures to be analysed we will consider a second example.

EXAMPLE 2

Determine the joint displacements, member forces, and reactions for the pin-jointed plane frame shown in Figure 3.

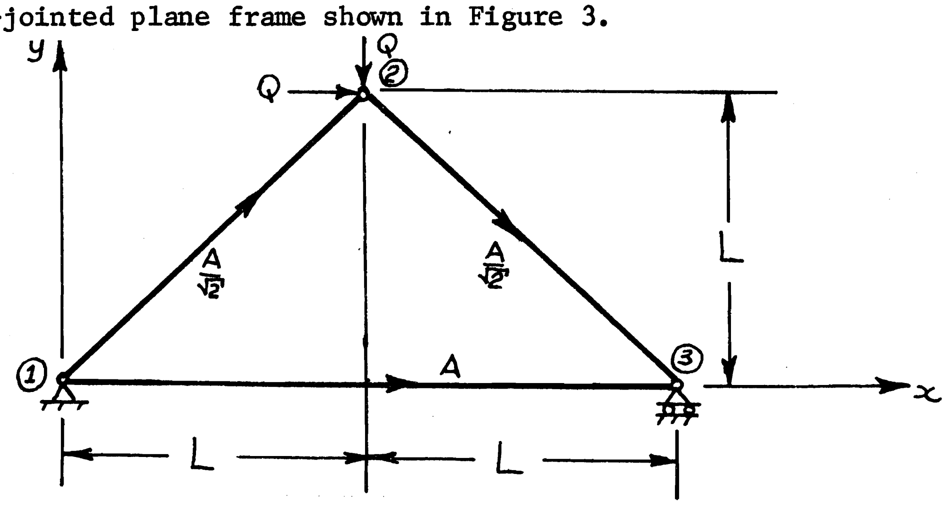


FIGURE 3

2.12 APPLICATIONS - SIMPLE TRIANGULATED PIN-JOINTED TRUSS (Contd.)

SOLUTION

Observations of Figure 3 immediately indicates that a problem confronts us. In the previous example all the members were co-linear, i.e. they were all in member co-ordinates. In this case we have members of varying orientation and yet to obtain a meaningful solution we require that all members be referred to a common co-ordinate system which in general is referred to as a frame co-ordinate system. A solution to this problem could be obtained in a number of ways. To keep the analysis compatible with that of the previous example we will proceed in the following manner. Consider the isolated member shown in Figure 4.

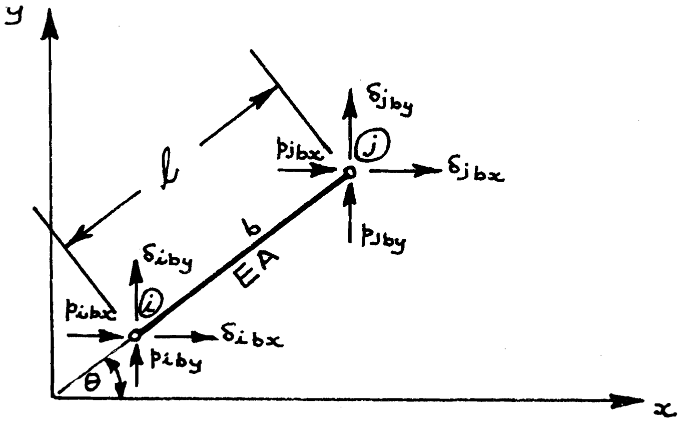


FIGURE 4

From equil. considerations

$$\Sigma F_x = 0 = p_{ibx} + p_{jbx}$$

$$\therefore \; p_{ibx} = -p_{jbx}$$

$$\Sigma F_y = 0 = p_{iby} + p_{jby}$$

$$\therefore \; p_{iby} = -p_{jby}$$

Compatibility conditions dictate member deformations such that :

$$\text{Member deformation} = \Delta\ell = \delta_{jbx}\cos\theta + \delta_{jby}\sin\theta - (\delta_{ibx}\cos\theta + \delta_{iby}\sin\theta)$$

$$\text{Force in the bar} = p_b = \frac{\Delta\ell}{\ell}.E.A.$$

Also

$$p_{jbx} = p_b\cos\theta; \quad p_{jby} = p_b\sin\theta$$

Hence

$$p_{jbx} = \frac{\Delta\ell}{\ell}.E.A.\cos\theta$$

$$p_{jby} = \frac{\Delta\ell}{\ell}.E.A.\sin\theta$$

Hence to satisfy both equilibrium and compatibility the following relationships must exist

$$p_{ibx} = \frac{EA}{\ell}\left[(\delta_{ibx}\cos^2\theta + \delta_{iby}\sin\theta\cos\theta)-(\delta_{jbx}\cos^2\theta + \delta_{jby}\sin\theta\cos\theta)\right]$$

$$p_{iby} = \frac{EA}{\ell}\left[(\delta_{ibx}\sin\theta\cos\theta + \delta_{iby}\sin^2\theta)-(\delta_{jbx}\sin\theta\cos\theta + \delta_{jby}\sin^2\theta)\right]$$

$$p_{jbx} = \frac{EA}{\ell}\left[-(\delta_{ibx}\cos^2\theta + \delta_{iby}\cos\theta\sin\theta)+(\delta_{jbx}\cos^2\theta + \delta_{jby}\sin\theta\cos\theta)\right]$$

$$p_{jby} = \frac{EA}{\ell}\left[-(\delta_{ibx}\sin\theta\cos\theta + \delta_{iby}\sin^2\theta)+(\delta_{jbx}\sin\theta\cos\theta + \delta_{jby}\sin^2\theta)\right]$$

Hence in matrix form the equilibrium and compatibility relationships become

$$\left[\begin{array}{c} p_{ibx} \\ p_{iby} \\ \hline p_{jbx} \\ p_{jby} \end{array}\right] = \left[\begin{array}{cc|cc} \frac{EA}{\ell}\cos^2\theta & \frac{EA}{\ell}\sin\theta\cos\theta & -\frac{EA}{\ell}\cos^2\theta & -\frac{EA}{\ell}\sin\theta\cos\theta \\ \frac{EA}{\ell}\sin\theta\cos\theta & \frac{EA}{\ell}\sin^2\theta & -\frac{EA}{\ell}\sin\theta\cos\theta & -\frac{EA}{\ell}\sin^2\theta \\ \hline -\frac{EA}{\ell}\cos^2\theta & -\frac{EA}{\ell}\sin\theta\cos\theta & \frac{EA}{\ell}\cos^2\theta & \frac{EA}{\ell}\sin\theta\cos\theta \\ -\frac{EA}{\ell}\sin\theta\cos\theta & -\frac{EA}{\ell}\sin^2\theta & \frac{EA}{\ell}\sin\theta\cos\theta & \frac{EA}{\ell}\sin^2\theta \end{array}\right] \left[\begin{array}{c} d_{ibx} \\ d_{iby} \\ \hline d_{ibx} \\ d_{jby} \end{array}\right]$$

which relating to the previous example can be written once again as

$$\left[\begin{array}{c} p^1{}_{ib} \\ \hline p^1{}_{jb} \end{array}\right] = \left[\begin{array}{c|c} k11^1{}_b & k12^1{}_b \\ \hline k21^1{}_b & k22^1{}_b \end{array}\right] \left[\begin{array}{c} d^1{}_{ib} \\ \hline d^1{}_{jb} \end{array}\right] \quad \text{- - - - - - - - - - -(2.8)}$$

The dashes being used to distinguish between member and frame co-ordinates.

From the expanded form of the force - displacement relationships for a member we can make the following observations.

(1) Comparison with the equivalent expression of Example 1 shows that the only real difference is that the dimension of the sub-matrices have increased to 2 x 2 to account for the newly acquired degree of freedom at each end.

(2) It can also be seen that for a pin-jointed member

$$k11 = k22$$
$$k12 = k21 = -k11.$$

Continuing again with our example in a step by step procedure.

2.13 MEMBER STIFFNESS SUB-MATRICES

From the condition (2) above it can be seen that it is only necessary in this case to determine k11. This is best done for purpose of demonstration using a table as shown below.

Member	Cross-section	Length	cosθ	sinθ	k11
$\overline{1,2}$	$\frac{A}{\sqrt{2}}$	$\sqrt{2}\,L$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\begin{bmatrix} \frac{EA}{4L} & \frac{EA}{4L} \\ \frac{EA}{4L} & \frac{EA}{4L} \end{bmatrix}$
$\overline{2,3}$	$\frac{A}{\sqrt{2}}$	$\sqrt{2}\,L$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\begin{bmatrix} \frac{EA}{4L} & -\frac{EA}{4L} \\ -\frac{EA}{4L} & \frac{EA}{4L} \end{bmatrix}$
$\overline{1,3}$	A	2 L	1	0	$\begin{bmatrix} \frac{EA}{2L} & 0 \\ 0 & 0 \end{bmatrix}$

2.14 SETTING UP $\bar{K}$

This is done in exactly the same manner as for Example 1. In this case however, the member stiffness sub-matrices are 2 x 2 because of the extra degree of freedom introduced at each joint. Hence the dimensions of $\bar{K}$ is 6 x 6, again obtained from the product of the number of degrees of freedom multiplied by the number of joints. The relationship $\bar{K}d = \bar{w}$ then becomes

$$
\begin{array}{cc}
 & \overbrace{\qquad\qquad}^{1}\ \overbrace{\qquad\qquad}^{2}\ \overbrace{\qquad\qquad}^{3} \\
1\left\{\begin{array}{c}\\ \\ \\ \\ \end{array}\right. &
\left[\begin{array}{cc|cc|cc}
\frac{EA}{4L} & \frac{EA}{4L} & -\frac{EA}{4L} & -\frac{EA}{4L} & -\frac{EA}{2L} & 0 \\
\frac{EA}{2L} & 0 & & & & \\
\hline
\frac{EA}{4L} & \frac{EA}{4L} & -\frac{EA}{4L} & -\frac{EA}{4L} & 0 & 0 \\
0 & 0 & & & &
\end{array}\right]
\end{array}
\quad
\begin{bmatrix} d_{ix} \\ \hline d_{iy} \end{bmatrix}
=
\begin{bmatrix} \bar{r}_{ix} \\ \hline \bar{r}_{iy} \end{bmatrix}
$$

2.14 SETTING UP $\bar{K}$ (Contd.)

$$
\begin{array}{c c}
 & \begin{array}{cccccc} \multicolumn{2}{c}{\overbrace{\hspace{6em}}^{1}} & \multicolumn{2}{c}{\overbrace{\hspace{6em}}^{2}} & \multicolumn{2}{c}{\overbrace{\hspace{6em}}^{3}} \end{array} \\
\begin{array}{c} 2\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ 3\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} &
\left[\begin{array}{c|c|c|c|c|c}
-\frac{EA}{4L} & -\frac{EA}{4L} & \begin{matrix}\frac{EA}{4L} \\ \frac{EA}{4L}\end{matrix} & \begin{matrix}\frac{EA}{4L} \\ -\frac{EA}{4L}\end{matrix} & -\frac{EA}{4L} & \frac{EA}{4L} \\
\hline
-\frac{EA}{4L} & -\frac{EA}{4L} & \begin{matrix}\frac{EA}{4L} \\ -\frac{EA}{4L}\end{matrix} & \begin{matrix}\frac{EA}{4L} \\ \frac{EA}{4L}\end{matrix} & \frac{EA}{4L} & -\frac{EA}{4L} \\
\hline
-\frac{EA}{2L} & 0 & -\frac{EA}{4L} & \frac{EA}{4L} & \begin{matrix}\frac{EA}{4L} \\ \frac{EA}{2L}\end{matrix} & \begin{matrix}-\frac{EA}{4L} \\ 0\end{matrix} \\
\hline
0 & 0 & \frac{EA}{4L} & -\frac{EA}{4L} & \begin{matrix}-\frac{EA}{4L} \\ 0\end{matrix} & \begin{matrix}\frac{EA}{4L} \\ 0\end{matrix}
\end{array}\right]
\left[\begin{array}{c} d_{2x} \\ \hline d_{2y} \\ \hline d_{3x} \\ \hline d_{3y} \end{array}\right]
=
\left[\begin{array}{c} Q \\ \hline -Q \\ \hline 0 \\ \hline r_{3y} \end{array}\right]
\end{array}
$$

2.15 IMPLEMENTING CONSTRAINT CONDITIONS

This is done as before by suppressing the rows and columns of $\bar{K}$ associated with the restrained degree of freedom. Thus $\bar{K}$ is reduced to K and becomes

$$
\left[\begin{array}{c|c|c}
\frac{EA}{2L} & 0 & -\frac{EA}{4L} \\
\hline
0 & \frac{EA}{2L} & \frac{EA}{4L} \\
\hline
-\frac{EA}{4L} & \frac{EA}{4L} & \frac{3EA}{4L}
\end{array}\right]
\left[\begin{array}{c} d_{2x} \\ \hline d_{2y} \\ \hline d_{3x} \end{array}\right]
=
\left[\begin{array}{c} Q \\ \hline -Q \\ \hline 0 \end{array}\right]
$$

which can be rewritten as

$$
\left[\begin{array}{c|c|c}
2 & 0 & -1 \\
\hline
0 & 2 & 1 \\
\hline
-1 & 1 & 3
\end{array}\right]
\left[\begin{array}{c} d_{2x} \\ \hline d_{2y} \\ \hline d_{3x} \end{array}\right]
= \frac{4QL}{EA}
\left[\begin{array}{c} 1 \\ \hline -1 \\ \hline 0 \end{array}\right]
$$

2.16 JOINT DISPLACEMENTS

The joint displacements are obtained by solving the system of equations given in the matrix relationship of 2.15. This results in

2.16 JOINT DISPLACEMENTS (Contd.)

$$\begin{bmatrix} d_{2x} \\ d_{2y} \\ d_{3x} \end{bmatrix} = \frac{QL}{AE} \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$$

2.17 MEMBER FORCES

In this example the member forces cannot be determined directly from the second of Equations 2.8. The reason for this being that Equation 2.8 is written in terms of frame co-ordinates and to obtain the member forces it must be in terms of member co-ordinates. However, member end displacements should be left in frame co-ordinates for convenience of handling. In terms of Matrix Algebra this is a hybrid transformation in which the second of Equations 2.8 is pre-multiplied by the transpose of the transformation matrix.

In physical terms this simply amounts to resolving p_{jbx} and p_{jby} into the direction of the member being considered. Thus

$$p_{jb} = \begin{bmatrix} \frac{-EA}{L}\cos^3\theta - \frac{EA}{L}\cos\theta\sin^2\theta & \frac{-EA}{L}\cos^2\theta\sin\theta \; \frac{-EA}{L}\sin^3\theta \end{bmatrix} \begin{bmatrix} d_{ibx} \\ d_{iby} \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{EA}{L}\cos^3\theta + \frac{EA}{L}\cos\theta\sin^2\theta & \frac{EA}{L}\cos^2\theta\sin\theta \; \frac{EA}{L}\sin^3\theta \end{bmatrix} \begin{bmatrix} d_{jbx} \\ d_{jby} \end{bmatrix}$$

Applying the above relationship to member $\overline{23}$ as an example

$$p_{j\overline{23}} = \begin{bmatrix} \frac{-EA}{2\sqrt{2}\,L} & \frac{EA}{2\sqrt{2}\,L} \end{bmatrix} \begin{bmatrix} \frac{3QL}{AE} \\ \frac{-3QL}{AE} \end{bmatrix} + \begin{bmatrix} \frac{EA}{2\sqrt{2}\,L} & \frac{-EA}{\sqrt{2}\;2\,L} \end{bmatrix} \begin{bmatrix} \frac{2QL}{AE} \\ 0 \end{bmatrix}$$

$$= \frac{-3Q}{\sqrt{2}} + \left(-\frac{Q}{\sqrt{2}}\right)$$

$$\therefore \; p_{j\overline{23}} = \frac{-2Q}{\sqrt{2}} \quad \text{(compression)}.$$

By successive application of the member force equation to the other members of the frame gives

$$\begin{bmatrix} p_{\overline{12}} \\ p_{\overline{23}} \\ p_{\overline{13}} \end{bmatrix} = Q \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

2.18 REACTIONS

Once again the simplest way to determine the constraint reactions is by considering conditions of local equilibrium. It should be noted that reactions are always required in frame co-ordinates and whilst in Example 1 frame and member co-ordinates were synomous this is not the case now as we saw when determining member forces. Hence to apply the concept of summation of the negative of the member forces meeting at the constrained joint requires some little refinement. This refinement is simply a transformation back to frame co-ordinates. The reactive forces then become

$$\begin{bmatrix} R_{1x} \\ R_{1y} \\ R_{3x} \end{bmatrix} = \begin{bmatrix} -Q \\ 0 \\ Q \end{bmatrix}$$

2.19 CLOSURE TO EXAMPLE 2

The steps followed will not be restated since they are the same as for Example 1. However, there are a number of points worth noting. These are as follow :-

(1) Both $\bar{K}$ and K are symmetric. Further, if the elements of rows and columns of $\bar{K}$ are summed the result is zero.

(2) Both $\bar{K}$ and K are independent of the external loading applied to the structure. This is always the case when a linear analysis is being performed.

(3) The determinate of K is a positive number. This is very significant and will be discussed in more detail in the next section.

(4) Care must be taken to ensure that the correct co-ordinate system is being used when determining member forces and reactions.

(5) As the number of active degrees of freedom increase it becomes evident that the number of simultaneous equations to be solved also increases. Thus the efficient application of the method requires access to a computer.

(6) The number of simultaneous equations to be solved is equal to the number of active joints multiplied by their degrees of freedom.

2.20 SINGULARITY OF K

The significance of the determinant of K, mentioned in 2.19.3 will now be discussed in more detail. To simplify the discussion consider again the simple pin-jointed frame of Example 1 loaded and constrained as shown in Figure 4.

2.20 SINGULARITY OF K (Contd.)

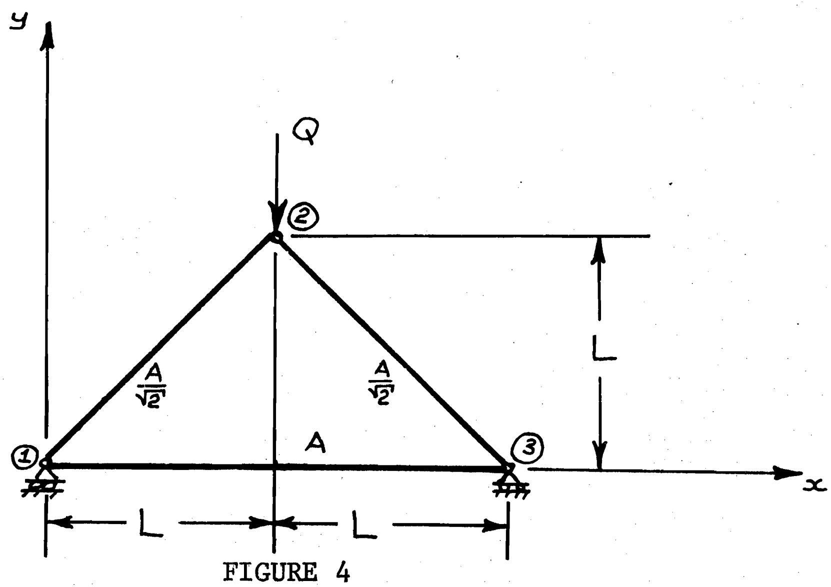


FIGURE 4

Since the topology and geometry of the above structure are the same as for Example 1 then $\bar{K}$ remains unchanged. However, because the method of constraining the structure is different, then K is changed and becomes

$$K = \begin{bmatrix} 3 & -1 & -1 & -2 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ -2 & -1 & 1 & 3 \end{bmatrix}$$

Magnitude of the determinant, found as follows results in

$$|K| = 3\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} +1\begin{bmatrix} -1 & 0 & -1 \\ -1 & 2 & 1 \\ -2 & 1 & 3 \end{bmatrix} -1\begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ -2 & -1 & 3 \end{bmatrix}$$

$$+ 2\begin{bmatrix} -1 & 2 & 0 \\ -1 & 0 & 2 \\ -2 & -1 & 1 \end{bmatrix}$$

$$= 3\left[2(5) - 1(2)\right] +1\left[-1(5) - 1(3)\right] -1\left[-1(1) \quad -2(-1) \quad -1(1)\right]$$

$$+ 2\left[-1(2) \quad -2(3)\right]$$

$$= 24 \quad - 8 - 0 - 16$$

$$= 24 - 24$$

$$|K| \quad = 0$$

2.20 SINGULARITY OF K (Contd.)

The above result could have been obtained much more easily using a simple row operation, which constitutes a theorem of determinant analysis. This row operation is carried out in the following manner :

Multiply the elements of Row 2 of K by unity and add these values to the corresponding elements of Row 1.

$$\begin{vmatrix} 2 & 1 & -1 & -3 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ -2 & -1 & 1 & 3 \end{vmatrix}$$

Doing similarly to the elements of Row 4 and adding to the elements of Row 1.

$$\begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -3 \\ -1 & 0 & 2 & 1 \\ -2 & -1 & 1 & 3 \end{vmatrix}$$

Since all the elements of Row 1 are zero then it follows that the magnitude of the determinate must be zero.

The concept of whether or not the determinant of K is positive, negative, or zero is of fundamental importance in structural analysis.

Returning to the case in point, i.e. when K = o, we have a situation in which it is impossible to get a set of displacements to satisfy conditions of equilibrium. This then means, in the jargon of Matrix Algebra, that the system is <u>inconsistent</u>, a condition characterised by the singularity (K = 0) of K. Singularity implies that one of two situations are in existance thus

(a) no solution exists

(b) there are infinitely many solutions.

In our case (b) applies because the structure can move to any position under a general load system.

<u>N.B.</u>

The fact that K is independent of the external loading is made very clear in this simple example. From observation it can be seen that for this example the structure is in equilibrium under the specified loading condition.

However, we also observe that the equilibrium is not stable because any slight horizontal force will set the structure in motion. Evidently, and fortunately, K does not recognise what we observe and is only concerned with having a sufficient number of suitable degrees of freedom restrained to render the structure stable externally. Further, and equally fortunately, K is oblivious to how many extra degrees of freedom are constrained beyond those required for statical determinacy.

The concept of singularity of K is also very conveniently used in determining the critical loads on structures together with their accompanying buckling modes. The subject of stability, being so complex and important in its own right, will not be discussed further here. The topic was only mentioned in an endeavour to stress the importance and significance of the structure stiffness matrix.

2.21 ILL-CONDITIONING

The topic of ill-conditioned systems of equations will not be pursued to any great depth at this point. However, it is felt to be of sufficient importance to at least warrant some discussion.

In physical terms ill-conditioning can be thought of in the following manner. Consider the relationship

$$Kd = w.$$

If the system is ill-conditioned then small changes in K will cause large changes in d, illustrating the non-dependence of conditioning on loading. Hence ill-conditioning is a function of the structure and if the equations of K are poorly conditioned then it can be assured that a floopy structure will result. Further, a well designed structure will always result in a structure stiffness matrix for which the system of equations are well conditioned. The problem of ill-conditioning is aptly described in the words of Dr. R.K. Livesley who said "ill-conditioning is not a disease but a symptom".

2.22 APPLICATIONS - SIMPLE RIGID JOINTED PLANE FRAME

Thus far only pin-jointed structures have been analysed. Since many structures transfer load by bending action as well as by axial forces a further example illustrating this class of structure will now be considered.

EXAMPLE 3

Determine the joint displacements, member forces, and reactions for the rigid jointed plane frame shown in Figure 5.

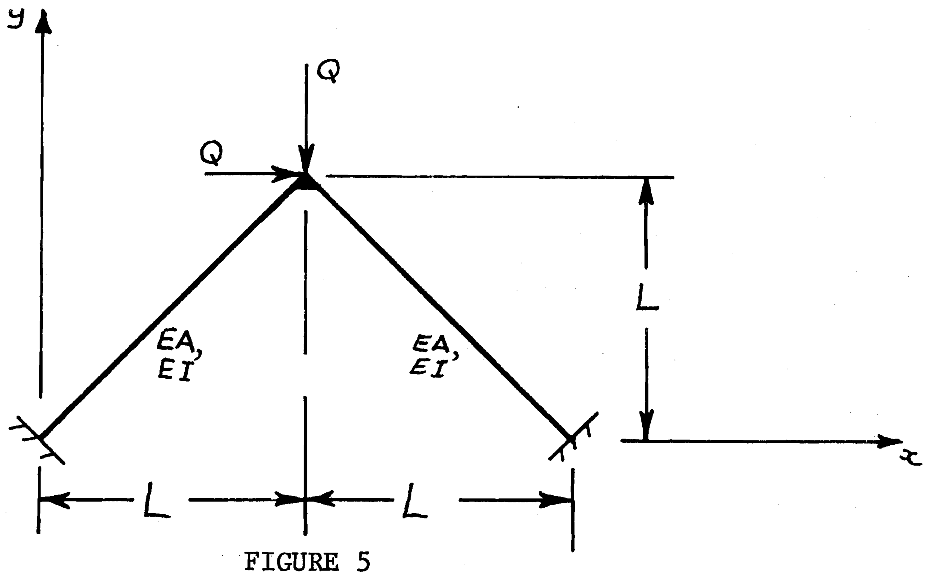


FIGURE 5

SOLUTION

It is evident that the member stiffness sub-matrices for a planar, rigid-jointed member will be different from those of a plane pin-jointed member. The member stiffness sub-matrices k11, k12, etc. can be found by using the following procedure.

A beam element, such as is being discussed in this example, will have three degrees of freedom at each of its ends as shown in Figure 6.

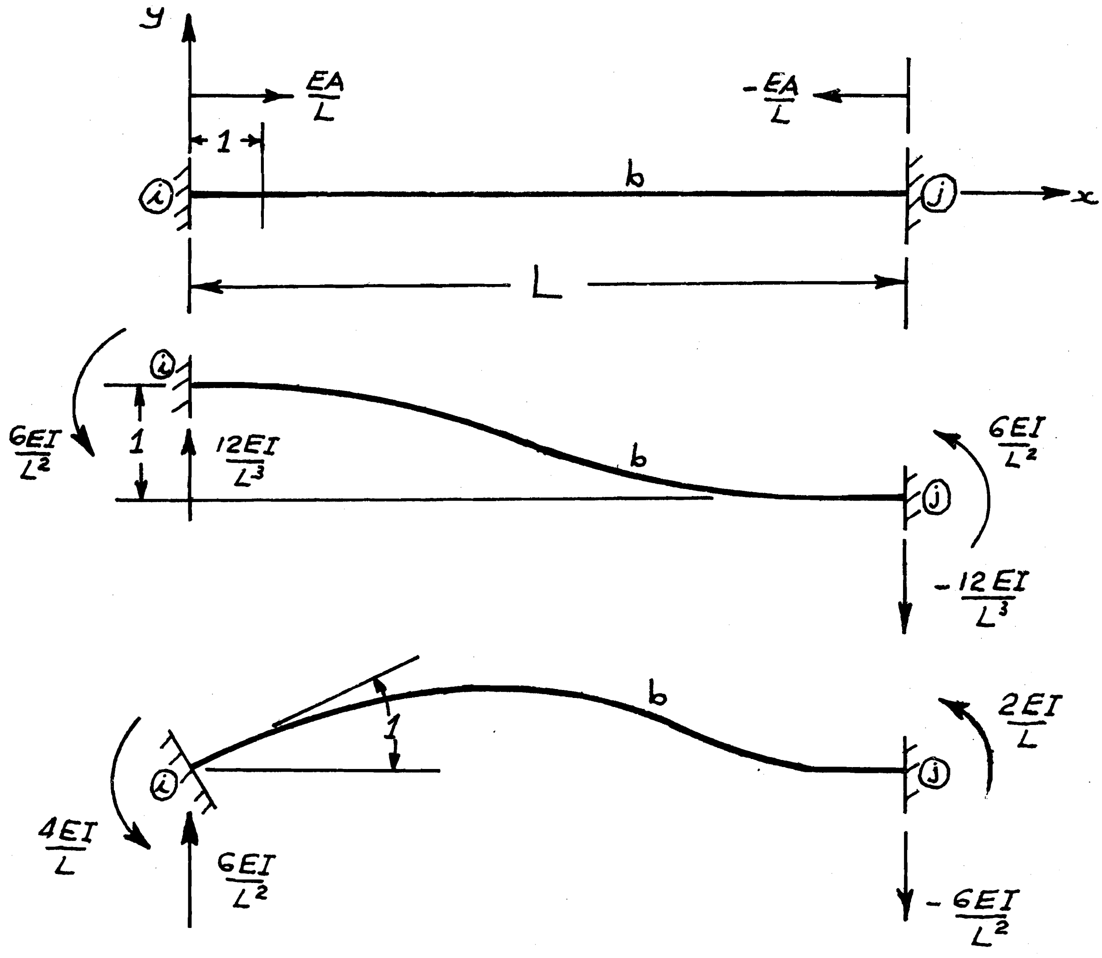


FIGURE 6.

2.22 APPLICATIONS - SIMPLE RIGID JOINTED PLANE FRAME (Contd.)

Since an "influence coefficient" for stiffnesses can be defined as the "forces developed at the joints due to a consistent set of unit displacements at a particular point", then, for an elastic analysis, combining the cases of Figure 6 gives

$$k11 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} ; \quad k12 = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \end{bmatrix}$$

$$k21 = k12^T = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} ; \quad k22 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

The elements of k11, k12, etc. being the forces associated with unit linear displacements in the x and y directions and a unit rotation about the z axis. Hence the vectors of member end displacements become

$$D_i = \begin{bmatrix} \delta_{ix} \\ \delta_{iy} \\ \theta_{iz} \end{bmatrix} ; \quad D_j = \begin{bmatrix} \delta_{jx} \\ \delta_{jy} \\ \theta_{jz} \end{bmatrix}$$

The above stiffness sub-matrices are associated with a system of member co-ordinates and since the members of the example are sloping then k11, k12 etc. must be transformed to a frame co-ordinate system. This can be done most simply by using matrix algebra. Because the process involved is simple and of fundamental importance to the application of the Stiffness Method the matrix manipultions used will be performed.

Consider again the relationship

$$k \; d = w$$

This can be written as

$$k \, I \, d = w$$

where I is the identity Matrix.

2.22 APPLICATIONS - SIMPLE RIGID JOINTED PLANE FRAME (Contd.)

Premultiply the second of the above equations by a transformation matrix T and replacing I by $T^T T$ gives

$$\boxed{T\,k\,T^T T\,d \;=\; T\,w} \qquad \text{----------(2.9)}$$

It should be noted that Equation 2.9 applies only to an orthogonal transformation or in other words $T^T = T^{-1}$ only for an orthogonal transformation.

For conformability T must be of the same order as k and is found as follows. Consider the isolated member (b) of Figure 7.

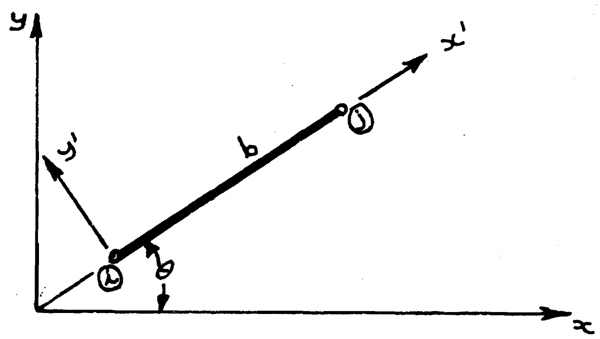


FIGURE 7

Resolving a unit vector in member co-ordinates i.e. in the x^1 direction, into frame co-ordinates gives the first column of T thus

$$\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

Following a similar procedure for the unit vector in the y^1 and z direction gives the second and third columns as

$$\begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} ; \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Combining the above gives the complete transformation matrix T as

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.23 MEMBER STIFFNESS SUB-MATRICES

In this case both k11 and k12 will have to be evaluated and then the fact that $k21 = k12^T$ and k22 = k11 with the off-diagonal elements negated will be utilised. Table 1 sets out the relevant information

Member	Cross-Section	Length	T	k11	k12
1,2	A	$\sqrt{2}\,L$	$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{EA}{2L} + \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{-6EI}{\sqrt{2}\,L^2} & \frac{6EI}{\sqrt{2}\,L^2} & \frac{4EI}{L} \end{bmatrix}$	$\begin{bmatrix} \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{6EI}{\sqrt{2}\,L^2} & \frac{-6EI}{\sqrt{2}\,L^2} & \frac{2EI}{L} \end{bmatrix}$
2,3	A	$\sqrt{2}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{EA}{2L} + \frac{6EI}{L^3} & \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{EA}{2L} + \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{6EI}{\sqrt{2}\,L^2} & \frac{6EI}{\sqrt{2}\,L^2} & \frac{4EI}{L} \end{bmatrix}$	$\begin{bmatrix} \frac{-EA}{2L} - \frac{-6EI}{L^3} & \frac{EA}{L} - \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{-6EI}{\sqrt{2}\,L^2} & \frac{-6EI}{\sqrt{2}\,L^2} & \frac{2EI}{L} \end{bmatrix}$

TABLE 1

The sub-matrices k21 and k22 are given by

$$k21_{\overline{12}} = \left[\begin{array}{cc|c} \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \hline \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \hline \frac{-6EI}{\sqrt{2}\,L^2} & \frac{6EI}{\sqrt{2}\,L^2} & \frac{2EI}{L} \end{array}\right]; \quad k22_{\overline{12}} = \left[\begin{array}{cc|c} \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{6EI}{\sqrt{2}\,L^2} \\ \hline \frac{-EA}{2L} + \frac{6EI}{L^3} & \frac{EA}{2L} + \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \hline \frac{6EI}{\sqrt{2}\,L^2} & \frac{-6EI}{\sqrt{2}\,L^2} & \frac{4EI}{L} \end{array}\right]$$

$$k21_{\overline{23}} = \left[\begin{array}{cc|c} \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \hline \frac{EA}{L} - \frac{6EI}{L^3} & \frac{-EA}{2L} - \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \hline \frac{6EI}{\sqrt{2}\,L^2} & \frac{6EI}{\sqrt{2}\,L^2} & \frac{2EI}{L} \end{array}\right]; \quad k22_{\overline{23}} = \left[\begin{array}{cc|c} \frac{EA}{2L} + \frac{6EI}{L^3} & \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \hline \frac{EA}{2L} - \frac{6EI}{L^3} & \frac{EA}{2L} + \frac{6EI}{L^3} & \frac{-6EI}{\sqrt{2}\,L^2} \\ \hline \frac{-6EI}{\sqrt{2}\,L^2} & \frac{-6EI}{\sqrt{2}\,L^2} & \frac{4EI}{L} \end{array}\right]$$

2.24 SETTING UP $\bar{K}$

This is done by following the same procedure as for the two previous examples.

$$\begin{array}{c} \\ 1 \\ \\ 2 \\ \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \qquad\qquad 2 \qquad\qquad & 3 \end{array} \\
\left[\begin{array}{c|ccc|c}
k11_{\overline{12}} & & k12_{\overline{12}} & & \\ \hline
k21_{\overline{21}} & \begin{matrix} \frac{EA}{2L} - \frac{6EI}{L^3} \\ \frac{EA}{2L} + \frac{6EI}{L^3} \end{matrix} & \begin{matrix} \frac{-EA}{2L} + \frac{6EI}{L^3} \\ \frac{-EA}{2L} + \frac{6EI}{L^3} \end{matrix} & \begin{matrix} \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{6EI}{\sqrt{2}\,L^2} \end{matrix} & k12_{\overline{23}} \\
 & \begin{matrix} \frac{-EA}{2L} + \frac{6EI}{L^3} \\ \frac{-EA}{2L} + \frac{6EI}{L^3} \end{matrix} & \begin{matrix} \frac{EA}{2L} + \frac{6EI}{L^3} \\ \frac{EA}{2L} + \frac{6EI}{L^3} \end{matrix} & \begin{matrix} \frac{-6EI}{\sqrt{2}\,L^2} \\ \frac{6EI}{\sqrt{2}\,L^2} \end{matrix} & \\
 & \begin{matrix} \frac{6EI}{\sqrt{2}\,L^2} \\ \frac{6EI}{\sqrt{2}\,L^2} \end{matrix} & \begin{matrix} \frac{-6EI}{\sqrt{2}\,L^2} \\ \frac{6EI}{\sqrt{2}\,L^2} \end{matrix} & \begin{matrix} \frac{4EI}{L} \\ \frac{4EI}{L} \end{matrix} & \\ \hline
 & & k21_{\overline{23}} & & k22_{\overline{23}}
\end{array}\right] \end{array}
\left[\begin{array}{c} d_{1x} \\ d_{1y} \\ \theta_{1z} \\ \hline d_{2x} \\ d_{2y} \\ \theta_{2z} \\ \hline d_{3x} \\ d_{3y} \\ \theta_{3z} \end{array}\right] = \left[\begin{array}{c} \bar{r}_{1x} \\ \bar{r}_{1y} \\ r_{1z} \\ \hline Q \\ -Q \\ 0 \\ \hline \bar{r}_{3x} \\ \bar{r}_{3y} \\ \bar{r}_{3z} \end{array}\right]$$

2.25 IMPLEMENTING CONSTRAINT CONDITIONS

Suppressing the restrained degrees of freedom reduces $\bar{K}$ to K thus

$$\left[\begin{array}{c|c|c} A & -A + \frac{12I}{L^2} & \frac{12I}{\sqrt{2}\,L} \\ \hline -A - \frac{12I}{L^2} & A + \frac{12I}{L^2} & 0 \\ \hline \frac{12I}{\sqrt{2}\,L} & 0 & 8I \end{array}\right] \left[\begin{array}{c} d_{2x} \\ \hline d_{2y} \\ \hline \theta_{2z} \end{array}\right] = \frac{QL}{E} \left[\begin{array}{c} 1 \\ \hline -1 \\ \hline 0 \end{array}\right]$$

To simplify the solution of the above system of equations it will be assumed that $\frac{12I}{L^2}$ can be neglected.

The above relationship then becomes

$$\begin{bmatrix} A & -A & \frac{6\sqrt{2}\,I}{L} \\ -A & A & 0 \\ \frac{6\sqrt{2}\,I}{L} & 0 & 8I \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{2y} \\ \theta_{21z} \end{bmatrix} = \frac{QL}{E} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

2.26 JOINT DISPLACEMENTS

Solving the above system of equations for the joint displacements

$$\begin{bmatrix} d_{2x} \\ d_{2y} \\ \theta_{2z} \end{bmatrix} = \frac{QL}{E} \begin{bmatrix} \frac{8L^2}{2I} \\ \frac{4AL^2}{I} - \frac{1}{A} \\ \frac{-L}{6\sqrt{2}\,I} \end{bmatrix}$$

2.27 MEMBER FORCES

In this example the member forces will not be determined implicitly, however a brief discussion concerning their evaluation is considered necessary.

As always the member forces are determined in member co-ordinates. In a frame consisting of beam elements however, it is not simply a matter of determining p_{jb} and then negating this force to find p_{ib}. Using Equation 2.9 and performing a hybrid transformation to determine the member forces in member co-ordinates results in

$$\left.\begin{aligned} p_{ib} &= k11T^TTd_{ib} + k12T^TTd_{jb} \\ p_{jb} &= k21T^TTd_{ib} + k22T^TTd_{jb} \end{aligned}\right\} \text{- - - - - - - - - - -(2.10)}$$

2.27 MEMBER FORCES (Contd.)

It should be noted that compatibility relative to the frame co-ordinate system is satisfied by Equation 2.10 because Td_{ib} and Td_{jb} are joint displacements in frame co-ordinates.

To obtain a relationship between member end forces consider the beam element (b) of Figure 8.

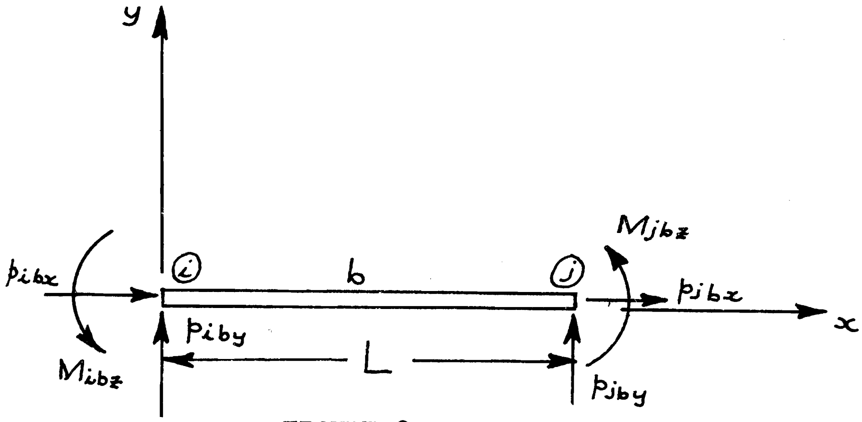


FIGURE 8

from equilibrium considerations

$$p_{ibx} + p_{jbx} = 0$$

$$p_{iby} + p_{jby} = 0$$

$$M_{ibz} + p_{jby}.L + M_{jbz} = 0.$$

or writing the above in matrix form

$$\begin{bmatrix} p_{ibx} \\ p_{iby} \\ M_{ibz} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L & 1 \end{bmatrix} \begin{bmatrix} p_{jbx} \\ p_{jby} \\ M_{jbz} \end{bmatrix} = 0$$

or more simply

$$\boxed{p_{ib} + H\ p_{jb} = 0} \quad \text{----------(2.11)}$$

where H is defined as the equilibrium matrix.

Hence if the member forces are known at one end of a member then by applying Equation 2.11 the forces at the other end can be determined, i.e.

$$p_{ib} = -H\ p_{jb}$$

OR

$$p_{jb} = -H^{-1}\ p_{ib}$$

where

$$H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -L & 1 \end{bmatrix}$$

2.28 REACTIONS

The frame reactions can be determined again from considerations of local equilibrium.

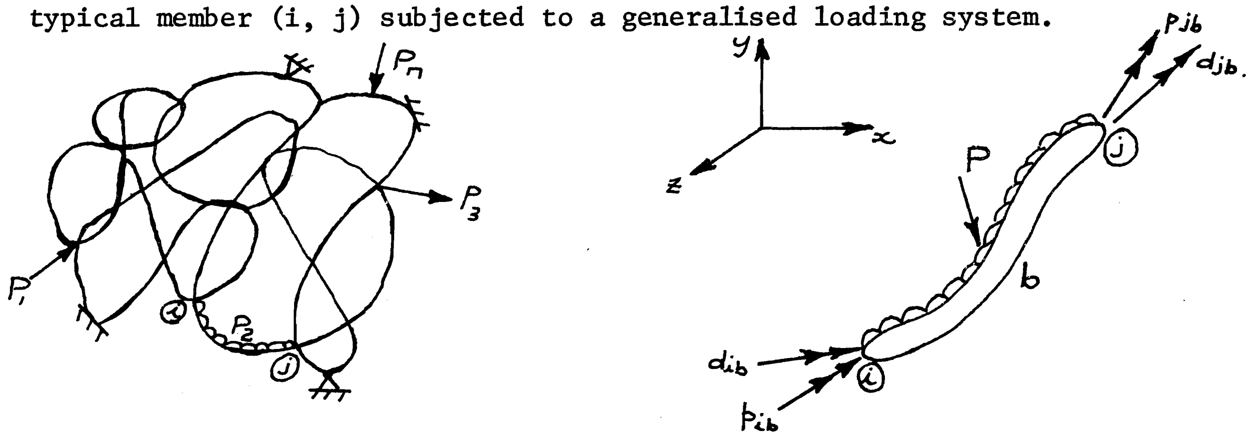
2.29 CLOSURE TO EXAMPLE 2

In this example the similarity of formulation with the two previous examples is very evident. Through the examples considered it should also be clear at this point that any structure consisting of discrete structural elements can be analysed provided the system of simultaneous equations can be solved. Clearly, if a space structure with 6 degrees of freedom per joint is to be analysed, not very many joints are required before the number of simultaneous equations to be handled becomes a major problem. Hence, it can be concluded that structural analysis, at least of discrete element structures, is no real problem, provided a computer of sufficient capacity is available.

2.30 OFF-JOINT LOADING

Thus far only the loading case associated with on-joint point loads has been considered. The usual situation in practice however, is that the loading is distributed along a structural member in some way or is applied as a concentrated load. Hence a need exists for considering the ways in which such loading types may be idealised.

In order to pursue the subject further consider the generalised skeletal structure shown in Figure 9. Also shown in the same figure is typical member (i, j) subjected to a generalised loading system.



NOOSHIN STRUCTURE

TYPICAL MEMBER

FIGURE 9

The typical member of Figure 9 is free to displace at each of its ends i and j. Suppose now that the member is fully fixed at each end as shown in Figure 10.

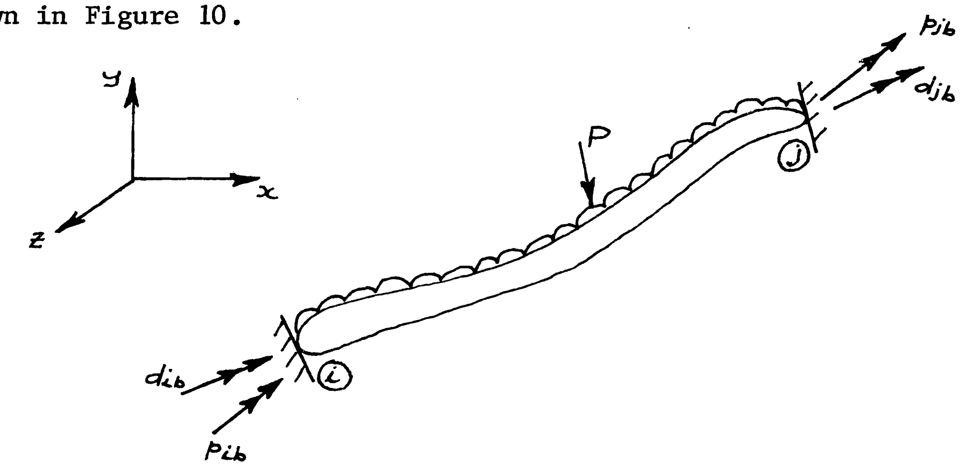


FIGURE 10

The fixed end reactive forces for the fixed ended member b are then given by

$$R_{ib} = \begin{bmatrix} R_{ibx} \\ R_{iby} \\ R_{ibz} \\ M_{ibx} \\ M_{iby} \\ M_{ibz} \end{bmatrix} \quad ; \quad R_{jb} = \begin{bmatrix} R_{jbx} \\ R_{jby} \\ R_{jbz} \\ M_{jbx} \\ M_{jby} \\ M_{jbz} \end{bmatrix}$$

Considering now the two cases of Figures 9 and 10, then

(1) For joints loaded and displacing

The familiar force - displacement relationships

$$p_{ib} = k11_b d_{ib} + k12_b d_{jb}$$

$$p_{jb} = k21_b d_{ib} + k22_b d_{jb}$$

must still hold.

(2) For off-joint loads with no joint displacement

This is the equally familiar case which is usually associated with the Method of Moment-Distribution and results in the reactive forces vectors R_{ib} and R_{jb} associated with Figure 10.

The generalised force-displacement relationships for any member such as (b) of Figure 10 can be found by superposition thus

2.30 OFF-JOINT LOADING (Contd.)

$$\left.\begin{aligned} p_{ib} &= k11_b\ d_{ib} + k12_b\ d_{jb} + R_{ib} \\ p_{jb} &= k12_b\ d_{ib} + k22_b\ d_{jb} + R_{jb} \end{aligned}\right\} \quad \text{--------- -(2.12)}$$

Equations 2.12 will be immediately recognised as the Slope - Deflection relationships.

Compatibility conditions are satisfied simply by replacing member end displacements by joint displacements.

To satisfy equilibrium conditions consider first of all the case where the loads are applied at the joints of the structure. Equilibrium of the i th joint is satisfied by

$$\Sigma p_{ib} = w_i$$

Where off-joint loads exist then the equilibrium relationship for the i th joint is given by

$$\Sigma p_{ib} = w_i - \Sigma R_{ib}$$

The above relationship merely states that the external loading at the i th joint consists of any external applied joint loads together with the negative of the reactive forces at that particular joint.

2.30 SELF-STRAINING SYSTEMS

In certain circumstances complex elastic structures become strained without the application of external loads. The three most common causes of such straining are

(1) Temperature effects.

(2) Lack of fit.

(3) Displacement effects.

which are usually associated with support settlement. Each of the above will now be considered in turn.

2.31 TEMPERATURE EFFECTS

Consider again the typical member b of Figure 11 isolated from a generalised structure.

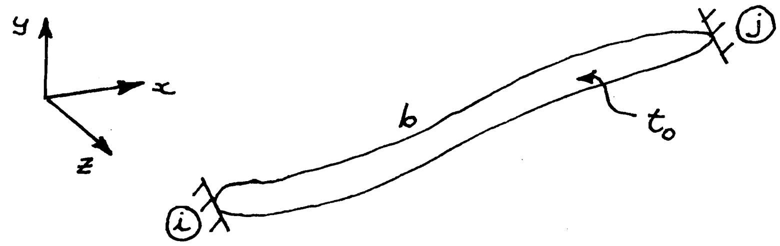


FIGURE 11

2.31 TEMPERATURE EFFECTS (Contd.)

For simplicity assume the member to be subjected to a uniform temperature t_o. The resulting change in temperature causes a set of reactive forces R_{ib} and R_{jb} to be set-up in the member similar to those due to off-joint loading.

When the member is uneffected by temperature change but the ends i and j are moving then

$$p_{ib} = k11_b d_{ib} + k12_b d_{jb}$$

$$p_{jb} = k21_b d_{ib} + k22_b d_{jb}$$

Conditions of equilibrium at each joint not effected by the temperature change is given by

$$\Sigma p_{ib} = 0.$$

Equilibrium conditions for joints associated with members subjected to a temperature variation dictates that

$$\Sigma p_{ib} = -\Sigma R_{ib}.$$

Hence the fundamental force-displacement relationship for the structure becomes

$$K\ d = w = -\Sigma R_{ib}$$

The above relationship is the same as the one which would be obtained for a normal structural analysis considering on-joint loading except that in this case the load vector consists entirely of the negative of member end reactive forces.

2.32 LACK OF FIT

Figure 12 shows the same member b. However this time it is assumed to suffer a fabrication imperfection λ.

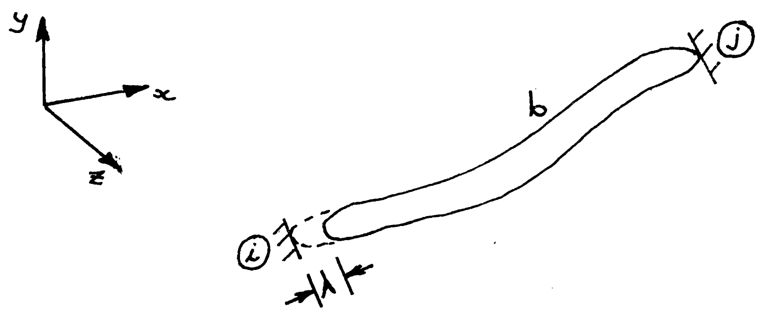


FIGURE 12

Because of the imperfection λ the member has to be deformed to allow it to fit. The resulting straining causes a set of reactive forces R_{ib} and R_{jb} to be induced into the member.

2.32 LACK OF FIT (Contd.)

This is a similar problem to the one just considered and evidently the member force-displacement relationship will be given by

$$p_{ib} = k11_b d_{ib} + k12_b d_{jb} + R_{ib}$$

$$p_{jb} = k11_b d_{ib} + k22_b d_{jb} + R_{jb}$$

The remainder of the analysis is the same as for 2.34.

2.33 DISPLACEMENT EFFECTS

In this case there is something of a variation of the theme. It is assumed that the structure is subjected to no external loading, however there has been a known measurable joint displacement. Suppose this movement is represented by

$$\delta_s = g$$

where: δ_s = a vector of the joint displacement, its order depending on the number of degrees of freedom of the joint.

g = known values of the displacements.

Setting up the primary stiffness matrix gives

$$\boxed{\bar{K}\, d = \bar{w}} \quad \text{- - - - - - - - - - -} \quad (2.13)$$

where: $\bar{w}$ = a vector containing the reactions.

Equation 2.13 can now be partitioned in the following manner

$$\left[\begin{array}{c|c|c} \bar{K}_{tt} & \bar{k}_{ts} & \bar{K}_{tu} \\ \hline \bar{K}_{st} & \bar{k}_{ss} & \bar{K}_{su} \\ \hline \bar{K}_{ut} & \bar{k}_{us} & \bar{K}_{uu} \end{array}\right] \left[\begin{array}{c} d_t \\ \hline \delta_s \\ \hline d_u \end{array}\right] = \left[\begin{array}{c} \bar{w}_t \\ \hline \bar{w}_s \\ \hline \bar{w}_u \end{array}\right]$$

Carrying out the matrix multiplication gives

$$\bar{K}_{tt} d_t + \bar{k}_{ts} \delta_s + \bar{K}_{tu} d_u = \bar{w}_t$$

$$\bar{K}_{st} d_t + \bar{k}_{ss} \delta_s + \bar{K}_{su} d_u = \bar{w}_s$$

$$\bar{K}_{ut} d_t + \bar{k}_{us} \delta_s + \bar{K}_{uu} d_u = \bar{w}_u$$

Substituting the measured values of g for δ_s in the above

$$\begin{array}{lllll}
\bar{K}_{st}\, d_t + & \bar{k}_{ts}\, g & + \bar{K}_{tu} d_u & = & \bar{w}_t \\
\bar{K}_{st}\, d_t + & \bar{k}_{ss}\, g & + \bar{K}_{su} d_u & = & \bar{w}_s \\
\bar{K}_{ut}\, d_t + & \bar{k}_{us}\, g & + \bar{K}_{uu} d_u & = & \bar{w}_u
\end{array}$$

Transferring the known values contained in the box to the right hand side of the equation results in

$$\begin{array}{lllll}
\bar{K}_{tt} d_t & + \bar{K}_{tu} d_u & = & \bar{w}_t & - \bar{k}_{ts} g \\
\bar{K}_{st} d_t & + \bar{K}_{su} d_u & = & \bar{w}_s & - \bar{k}_{ss} g \\
\bar{K}_{ut} d_t & + \bar{K}_{uu} d_u & = & \bar{w}_u & - \bar{k}_{us} g.
\end{array}$$

In the above system of equations there is one more equation than there is unknowns. Therefore, one of the equations is linearly dependent and the problem is to decide which one. The second equation is required to satisfy the equilibrium and joint compatibility. However, this has already been satisfied by the known relationship transferred to the right hand side. Therefore the second equation can be removed from the system giving

$$\left[\begin{array}{c|c} K_{tt} & K_{tu} \\ \hline K_{ut} & K_{uu} \end{array}\right] \left[\begin{array}{c} d_t \\ \hline d_u \end{array}\right] = \left[\begin{array}{cc} w_t & -k_{ts} g \\ \hline w_u & -k_{us} g \end{array}\right]$$

If there are a number of such displacements a similar procedure is followed for each one.

2.34 SEMI-SKELETAL STRUCTURES

The arguments developed so far have been concerned only with two-legged discrete elements.

Consider now the structure of Figure 13 in which the members consist of a continuum connected at discrete joints. The figure also shows a typical element (b) of the structure.

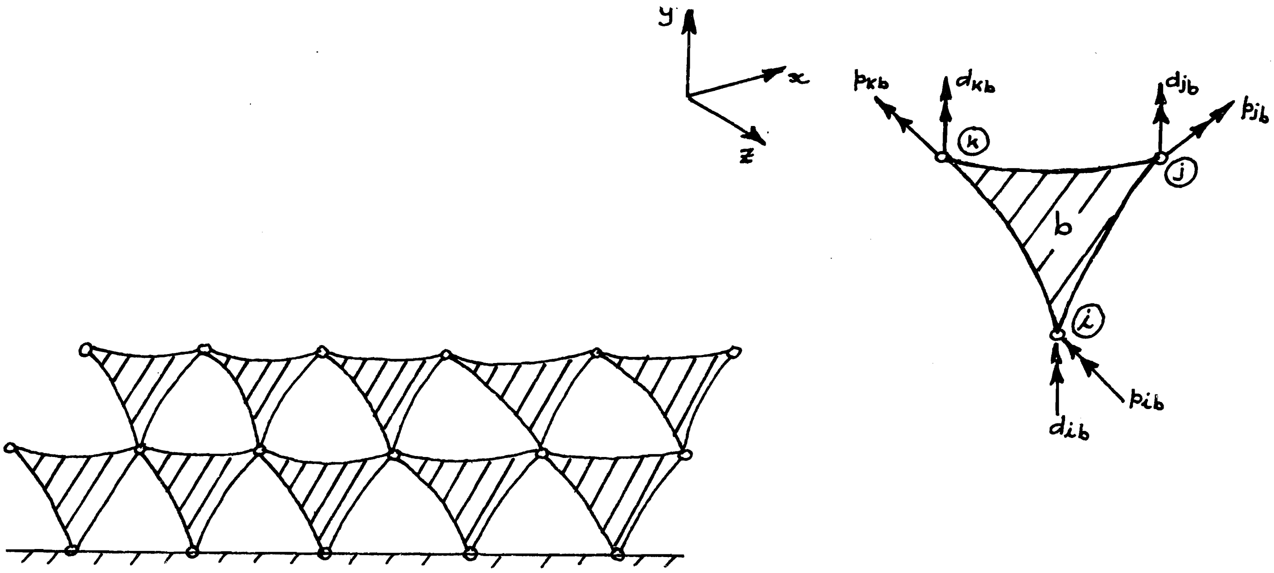


FIGURE 13

Equilibrium and compatibility must again be satisfied at all levels of the structure. Also forces and displacements cannot be independent and are given by

$$\left.\begin{aligned} p_{ib} &= k11_b d_{ib} + k12_b d_{jb} + k13_b d_{kb} + R_{jb} \\ p_{jb} &= k21_b d_{ib} + k22_b d_{jb} + k23_b d_{kb} + R_{jb} \\ p_{kb} &= k31_b d_{ib} + k32_b d_{jb} + k33_b d_{kb} + R_{kb} \end{aligned}\right\} \qquad (2.14)$$

All the arguments developed previously are still valid and $\bar{K}$ can be set-up in a similar manner to that used for the previous three examples. Constraint conditions can then be implemented as before thus reducing $\bar{K}$ to K and the system of equations solved for the unknown displacements. Member forces can be determined from the Equations 2.14 if self-straining of the system is also present. Local equilibrium conditions are then used to determine reactions.

2.35 STIFFNESS MATRICES

Only a very limited range of member stiffness matrices have been considered. Both the pin-jointed planar element and the rigid-jointed beam element can easily be generalised to their equivalent spatial members. It should be realised that by using these simple members many complex structural problems can be solved. Appendix A of this book contains a detailed treatment of the development of the stiffness matrix for a coplanar beam element. The difficulty arises when a discrete element is used which deviates from the norm. If it is necessary that such an element be incorporated in a structure then it is suggested that a thorough literature search be carried out in a hope that its stiffness matrix has already been developed.

2.35 STIFFNESS MATRICES (Contd.)

Should it not be available in the literature the problem would best be handled by passing it on to the Academic Fraternity.

2.37 CONCLUSION

By choice many areas have not been considered in this chapter. Some of the more important topics not treated are

(a) Implementing the imposition of displacements.
(b) Super-member and super-joint concepts.
(c) Non-conformable constraints.
(d) Computational techniques.
(e) Symmetry.
(f) Optimisation.

Obviously a knowledge of all of the above is important if full use is to be made in the application of the Stiffness Method. However, it is unnecessary to be familiar with the above topics for a complete conceptual understanding of the method.

In conclusion, it is felt necessary to point out that although the Stiffness Method is admirably suited to automatic computational techniques it is absolutely essential that some simple problems be attempted manually to become completely conversant with the mechanics of the steps involved.

TUTORIAL PROBLEMS

THE STIFFNESS METHOD

QUESTION 1

For the pin-ended member (b) shown in Figure 1 develop a relationship between member end forces and displacements in global coordinates, which satisfies both joint equilibrium and compatibility.

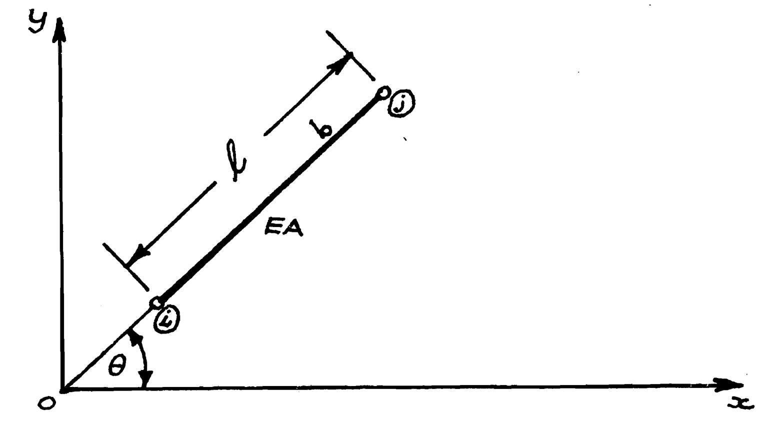


FIGURE 1

QUESTION 2

(a) State the levels at which equilibrium and compatibility must be satisfied within a structure.

(b) Explain briefly how this is done at any one level of the structure.

(c) Given the relationship $\bar{K}d = \bar{w}$ show how constraint conditions are implemented to reduce the above relationship to $Kd = w$.

QUESTION 3

For the simple Hookean Spring system loaded as shown in Figure 2, determine

(a) the joint displacements

(b) the spring forces

(c) the reactions.

FIGURE 2

QUESTION 4

Determine the joint displacements, member forces, and reactions for the plane pin-jointed frame shown in Figure 3.

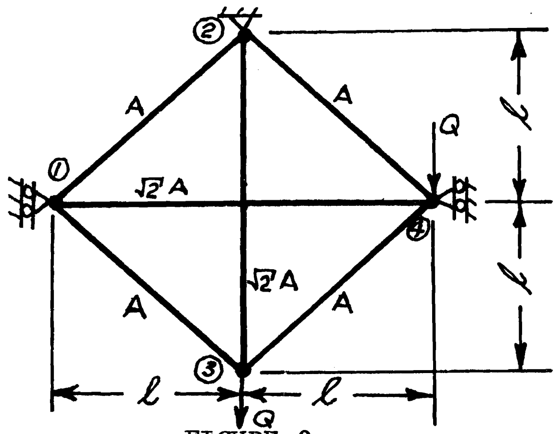


FIGURE 3

QUESTION 5

Develop the primary stiffness matrix for the structure shown in Figure 4. All members are made of the same material.

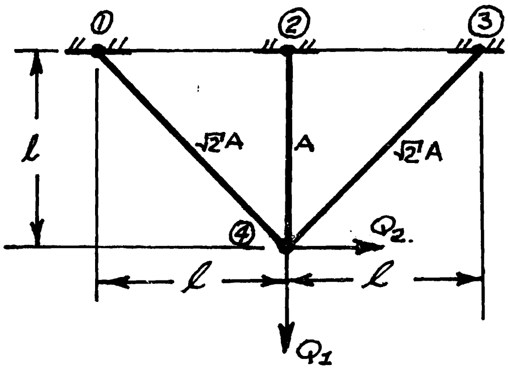


FIGURE 4

QUESTION 6

Determine the joint displacements and member forces for the structural spring system shown in Figure 5. No external loads have been applied to the system, however, joint 2 has been displaced 5 units to the right.

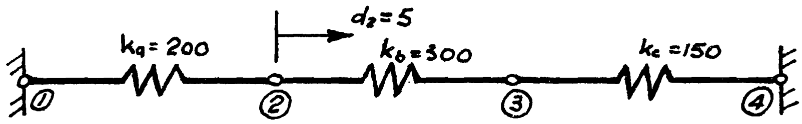


FIGURE 5

QUESTION 7

Find the joint displacements and member forces for the structures shown in Figures 6(a) and 6(b).

QUESTION 7 (Contd.)

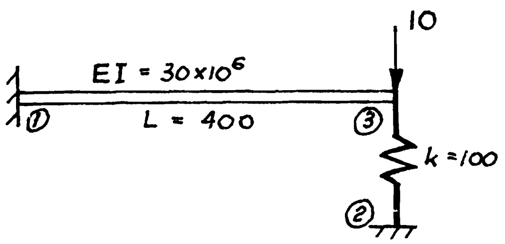


FIGURE 6(a)

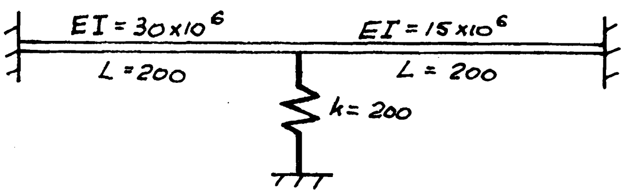


FIGURE 6(b)

QUESTION 8

In the structural system shown in Figure 7 member 2-3 is short by one (1) unit, determine

(a) the joint displacements

(b) the member forces.

Assume the lack of fit to be in member 3-4. What difference would it make to your answer if the lack of fit had been assumed in member 2-3.

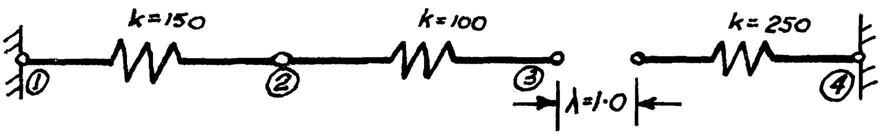


FIGURE 7

QUESTION 9

Member 2-3 of the structural system shown in Figure 8 is subjected to a contracting temperature differential of 30°. The length of the member is 50 units and the coefficient of expansion $\alpha = 0.001$. Determine

(a) the joint displacements

(b) the member forces.

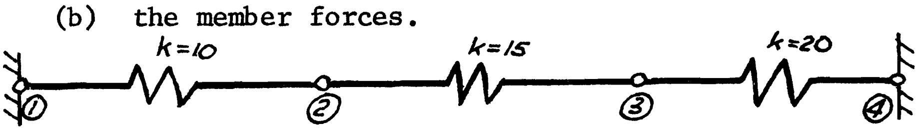


FIGURE 8

QUESTION 10

A member b of a pin-jointed skeletal structure is uniformily heated to cause an increase in the length of (b) by an amount αL_b. In general this will result in all of the degrees of freedom of the structure being activated. If (d) is the known displacement vector resulting at one of the joints of the structure show how the displacements at all of the other joints may be found analytically.

QUESTION 11

(a) Develop the stiffness sub-matrices k11, k12, k21, and k22 in member coordinates for the beam element shown in Figure 9(a).

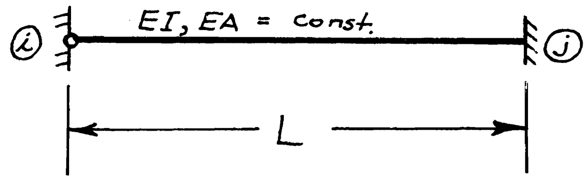


FIGURE 9(a)

(b) For the frame coordinate system x - y shown in Figure 9(b) evaluate the member stiffness sub-matrices for the beam element of (a) above, orientated as shown in Figure 9(b).

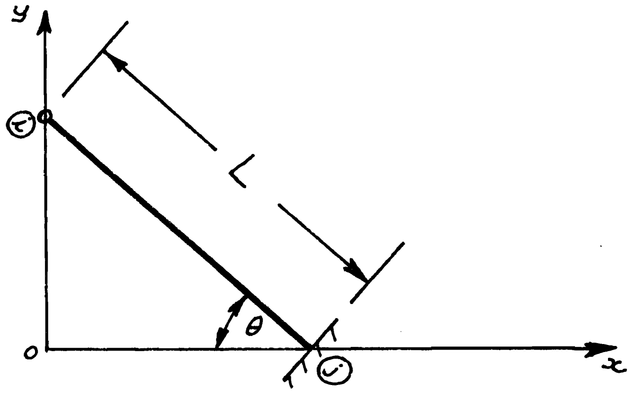


FIGURE 9(b)

QUESTION 12

Figure 10 shows a rigid jointed rectangular Portal Frame subjected to a horizontal point load applied to joint 2 and a u.d.l. applied to member 2-3.

(1) Determine, using the Stiffness Method

- (a) Joint displacements
- (b) Member forces
- (c) Reactions.

Neglect axial force effects as a first approximation.

(2) Using Moment Distribution determine the member forces for the frame of Figure 10.

(3) Using a plane frame computer package again determine the joint displacements, member forces, and reactions for the frame.

QUESTION 12 (Contd.)

(4) Compare the results obtained using 1, 2, and 3 above.

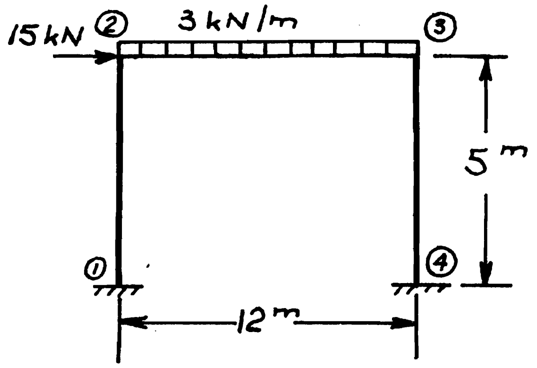


FIGURE 10

Data :

$E = 208 \times 10^6$ kN/m

$I_{cols} = 222 \times 10^6 \ mm^4$

$A_{cols} = 12{,}300 \ mm^2$

$I_{Beams} = 294 \times 10^6 \ mm^4$

$A_{Beams} = 8540 \ mm^2$

APPENDIX A

ELEMENT STIFFNESS MATRICES

When applying the Stiffness Method it is essential that the element stiffness matrices be known for the particular members which constitute the structure to be analysed.

A.1 DISPLACEMENT - FORCE (FLEXIBILITY) RELATIONSHIPS

These relationships will be developed for a planar beam element of uniform cross-section.

Consider the simply supported beam loaded as shown in Figure A.1, i.e. with end moments M_i and M_j applied.

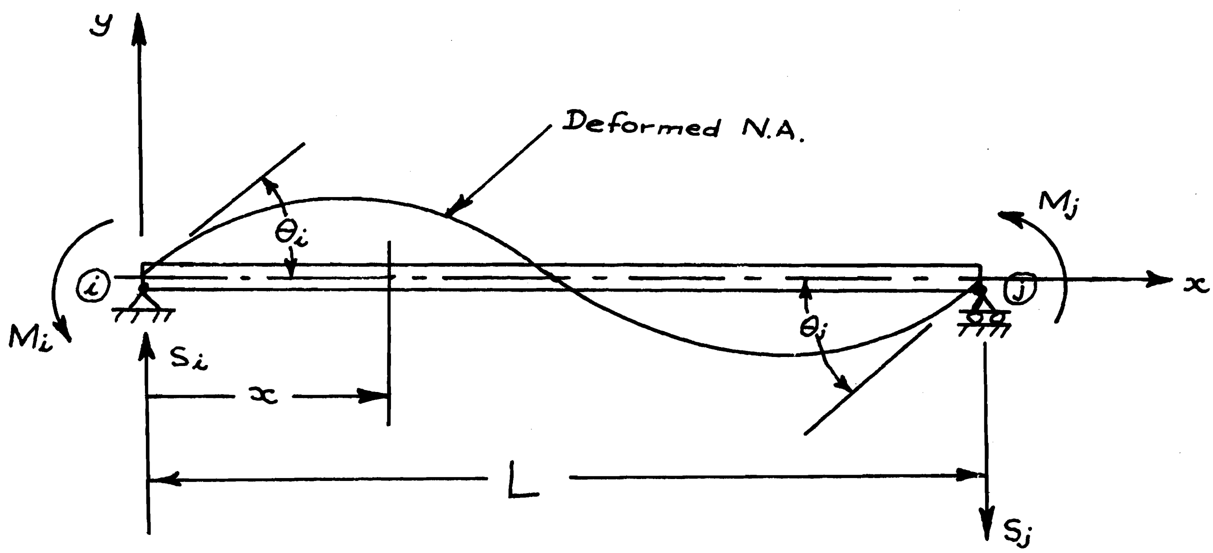


FIGURE A.1

From Castigliano's Second Theorem

$$\frac{\partial U}{\partial M_i} = \theta_i = \int_0^L \frac{M_x \left\{\frac{\partial M_x}{\partial M_i}\right\} dx}{EI} + \int_0^L \frac{\alpha S_x \left\{\frac{\partial S_x}{\partial M_i}\right\} dx}{GA}$$

$$\frac{\partial U}{\partial M_j} = \theta_j = \int_0^L \frac{M_x \left\{\frac{\partial M_x}{\partial M_j}\right\} dx}{EI} + \int_0^L \frac{\alpha S_x \left\{\frac{\partial S_x}{\partial M_j}\right\} d_x}{GA}$$

Substituting the relevant values into the above expressions gives

A.1 DISPLACEMENT - FORCE (FLEXIBILITY) RELATIONSHIPS (Contd.)

$$\theta_i = \frac{L}{6EI}(2M_i - M_j) + \frac{\alpha}{GAL}(M_i + M_j)$$

$$\theta_j = \frac{L}{6EI}(2M_j - M_i) + \frac{\alpha}{GAL}(M_i + M_j)$$

Putting :

$$k_o = \frac{6EI\alpha}{GAL^2}$$

$$\left.\begin{aligned} \theta_i &= \frac{L}{6EI}\left[M_i(2 + k_o) - M_j(1 - k_o)\right] \\ \theta_j &= \frac{L}{6EI}\left[M_j(2 + k_o) - M_i(1 - k_o)\right] \end{aligned}\right\} \text{- - - - - - - - - - - - - -(A.1)}$$

For no shear $k_o = 0$

$$\theta_i = \frac{L}{6EI}(2M_i - M_j)$$

$$\theta_j = \frac{L}{6EI}(2M_j - M_i)$$

OR

$$\begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = \frac{L}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} M_i \\ M_j \end{bmatrix}$$

A.2 RIGID BODY DISPLACEMENTS

Taking these effects into account through reference to Figure A.2 modifies Equations A.1 to Equations A.2

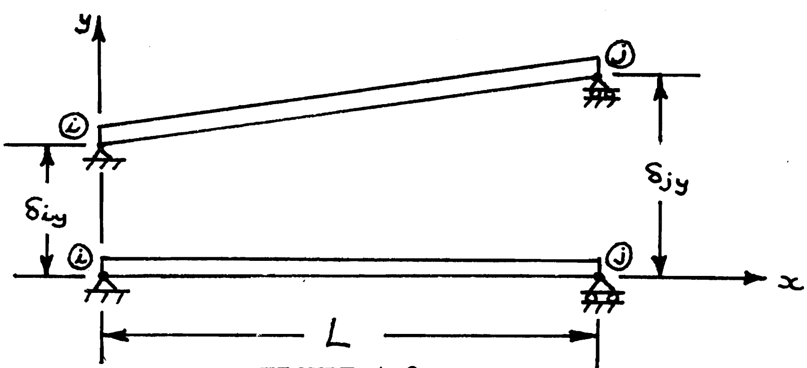


FIGURE A.2

$$\left.\begin{aligned} \theta_i &= \frac{L}{6EI}\left[M_i(2 + k_o) - M_j(1 - k_o) + (\delta_{jy} - \delta_{iy})/L\right] \\ \theta_j &= \frac{L}{6EI}\left[M_j(2 + k_o) - M_i(1 - k_o) + (\delta_{jy} - \delta_{iy})/L\right] \end{aligned}\right\} \text{- - - - - -(A.2)}$$

A.3 FORCE - DISPLACEMENT (STIFFNESS) RELATIONSHIPS

(a) Moments

Eliminating (M_j) from Equations A.2

$$\left.\begin{aligned} M_i &= \frac{2EI}{(1+2k_o)L}\left[(2+k_o)\,\theta_i + (1-k_o)\,\theta_j - 3(\delta_{jy} - \delta_{iy})/L\right] \\ M_j &= \frac{2EI}{(1+2k_o)L}\left[(1-k_o)\,\theta_i + (2+k_o)\,\theta_j - 3(\delta_{jy} - \delta_{iy})/L\right] \end{aligned}\right\} \text{- - -(A.3)}$$

Writing

$$\beta_1 = \frac{1-k_o}{1+2k_o}\;;\qquad \beta_2 = \frac{2+k_o}{1+2k_o}\;;\qquad \beta_3 = \frac{3}{1+2k_o}$$

$$\left.\begin{aligned} M_i &= \frac{2EI}{L}\left[\beta_2\,\theta_i + \beta_1\,\theta_j - \beta_3(\delta_{jy} - \delta_{iy})/L\right] \\ M_j &= \frac{2EI}{L}\left[\beta_1\,\theta_i + \beta_2\,\theta_2 - \beta_3(\delta_{jy} - \delta_{iy})/L\right] \end{aligned}\right\} \text{- - - - - - - - - - - -(A.4)}$$

For no shear $\beta_1 = 1$, $\beta_2 = 2$, and $\beta_3 = 3$

$$\left.\begin{aligned} M_i &= \frac{2EI}{L}\left[2\theta_i + \theta_j - 3(\delta_{jy} - \delta_{iy})/L\right] \\ M_j &= \frac{2EI}{L}\left[\theta_i + 2\theta_j - 3(\delta_{jy} - \delta_{iy})/L\right] \end{aligned}\right\} \text{- - - - - - - - - - - - - - (A.5)}$$

(b) Reactions

These are given by

$$S_i = P_{iy} = \frac{(M_i + M_j)}{L}$$

$$S_j = P_{jy} = \frac{-(M_i + M_j)}{L}$$

Substituting for M_i and M_j

$$\left.\begin{aligned} P_{iy} &= \frac{2EI}{L^2}\beta_3\left[\theta_i + \theta_j - 2(\delta_{jy} - \delta_{iy})/L\right] \\ P_{jy} &= \frac{-2EI}{L^2}\beta_3\left[\theta_i + \theta_j - 2(\delta_{jy} - \delta_{iy})/L\right] \end{aligned}\right\} \text{- - - - - - - - - - - -(A.6)}$$

(c) Axial Force Effects

These are given by

$$\left.\begin{aligned} P_{ix} &= \frac{EA}{L}(\delta_{ix} - \delta_{jx}) \\ P_{jx} &= \frac{-EA}{L}(\delta_{ix} - \delta_{jx}) \end{aligned}\right\} \text{- - - - - - - - - - - - - -(A.7)}$$

(d) Combined Effects

Combining the results of Equations A.4, A.6, and A.7 gives

$$\begin{bmatrix} P_{ix} \\ P_{iy} \\ M_i \end{bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{4EI\beta_3}{L^3} & \frac{2EI\beta_3}{L^2} \\ 0 & \frac{2EI\beta_3}{L^2} & \frac{2EI\beta_2}{L} \end{bmatrix} \begin{bmatrix} \delta_{ix} \\ \delta_{iy} \\ \theta_i \end{bmatrix} + \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-4EI\beta_3}{L^3} & \frac{2EI\beta_3}{L^2} \\ 0 & \frac{-2EI\beta_3}{L^2} & \frac{2EI\beta_1}{L} \end{bmatrix} \begin{bmatrix} \delta_{jx} \\ \delta_{jy} \\ \theta_j \end{bmatrix}$$

$$\begin{bmatrix} P_{jx} \\ P_{jy} \\ \theta_j \end{bmatrix} = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-4EI\beta_3}{L^3} & \frac{-2EI\beta_3}{L^2} \\ 0 & \frac{2EI\beta_3}{L^2} & \frac{2EI\beta_1}{L} \end{bmatrix} \begin{bmatrix} \delta_{ix} \\ \delta_{iy} \\ \theta_i \end{bmatrix} + \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{4EI\beta_3}{L^3} & \frac{-4EI\beta_3}{L^2} \\ 0 & \frac{-2EI\beta_3}{L^2} & \frac{2EI\beta_2}{L} \end{bmatrix} \begin{bmatrix} \delta_{jx} \\ \delta_{jy} \\ \theta_j \end{bmatrix}$$

which can be written in matrix notation thus

$$\left[\begin{array}{c} P_{ib} \\ \hline P_{jb} \end{array}\right] \quad \left[\begin{array}{c|c} k11 & k12 \\ \hline k21 & k22 \end{array}\right] \quad \left[\begin{array}{c} d_{ib} \\ \hline d_{jb} \end{array}\right]$$

NOTE :

(i) k11 and k22 are symmetric

(ii) k11 and k22 are identical

except off diagonal terms of k22 are negative.

(iii) $k21 = k12^T$

Neglecting shear

$$k11 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} ; \quad k12 = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} ; \quad k22 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Also $k21 = k12^T$.

A.4 PLANE GRILLAGE

The stiffness sub-matrices are as follow

$$k11 = \begin{bmatrix} \frac{GJ}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \; ; \; k12 = \begin{bmatrix} \frac{-GJ}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \end{bmatrix} \; ; \; k22 = \begin{bmatrix} \frac{GJ}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Also $k21 = k_{12}^T$.

A.5 RIGID-JOINTED SPACE STRUCTURE

In this case the number of degrees of freedom is doubled and k11 and k12 are given by

$$k11 = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}$$

$$k12 = \begin{bmatrix} \frac{-EA}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix}$$

A.5 RIGID-JOINTED SPACE STRUCTURE (Contd.)

k_{22} = k_{11} with the sign of the off diagonal term negated and

k_{21} = k_{12}^T.

A.6 CLOSURE TO STIFFNESS MATRICES

It should be noted that although only a limited range of element stiffness matrices have been considered in this appendix these relationships are used in practice to solve a wide range of skeletal structure problems.

APPENDIX B

STRUCTURAL DYNAMICS

The main purpose of this section is to introduce the topic of Structural Dynamics using a matrix approach. A general theory of free, undamped vibrations for structures with many degrees of freedom will be developed.

STIFFNESS FORMULATION OF THE EIGEN-PROBLEM

Consider the mass-spring system shown in Figure B.1

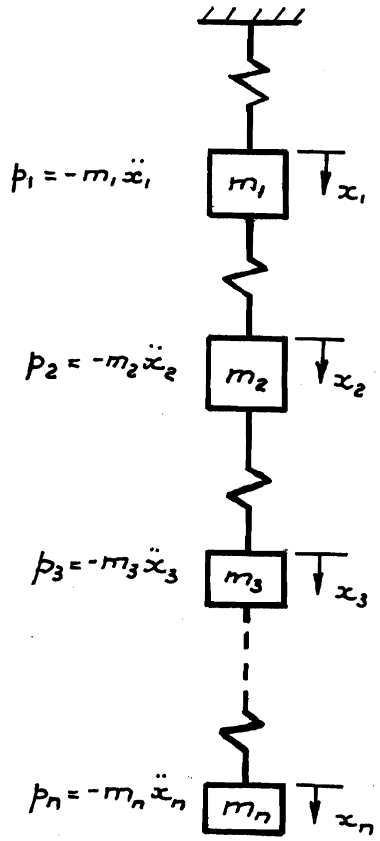


FIGURE B.1

The Equation of motion for the system of Figure B.1 can be expressed in matrix form as follows

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_n \end{bmatrix} = - \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & - & - & - & 0 \\ 0 & 0 & m_3 & - & - & 0 \\ \vdots & & \text{MASS} & & & \vdots \\ \vdots & & \text{MATRIX} & & & \vdots \\ 0 & - & - & - & - & m_n \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \vdots \\ \ddot{x}_n \end{bmatrix}$$

STIFFNESS FORMULATION OF THE EIGEN-PROBLEM (Contd.)

OR

$$\boxed{p = -M\ddot{x}} \quad \text{- - - - - - - - - -(B.1)}$$

where :

p is the forces vector

M is the mass matrix

$\ddot{x}$ is the acceleration vector.

The general force-displacement relationship can be written as

$$Kx = p$$

hence, Equation B.1 becomes

$$\boxed{Kx = -M\ddot{x}} \quad \text{- - - - - - - - - -(B.2)}$$

Unfortunately Equation B.2 is not yet in a usable form. To remove the time dependence associated with x assume

$$\left.\begin{array}{lllll} x_1 & = & X_1 & \sin & \omega t \\ x_2 & = & X_2 & \sin & \omega t \\ \vdots & & \vdots & & \\ x_n & = & X_n & \sin & \omega t \end{array}\right\} \quad \text{- - - - - - - - -(B.3)}$$

where

X_1, X_2, - - - - - - X_n is the <u>amplitude</u> of

vibration of masses and ω is the frequency of the system.

Differentiating Equation B.3 with respect to time twice results in

$$\ddot{x} = -\omega^2 \sin \omega t \; X.$$

Hence from Equation B.2 we get

$$\boxed{KX = \omega^2 MX} \quad \text{- - - - - - - - - -(B.4)}$$

There are two points to be noted concerning Equation B.4. These are

(i) The equation reduces the Equations of Motion from being <u>time</u> dependent to being <u>amplitude</u> dependent.

(ii) Equation B.4 is now obviously an eigenvalue problem in which there are n eigenvalues i.e. ω^2 or frequencies of vibration. Each frequency is associated with an eigenvector (mode vector).

To determine the eigenvalues of Equation B.4 we can rewrite it thus

$$\begin{bmatrix} K - \omega^2 M \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = 0 \quad \text{-------(B.5)}$$

Hence, for non-trivial solutions of Equation B.5 to exist

$$\begin{vmatrix} K - \omega^2 M \end{vmatrix} = 0$$

resulting in a set of ω^2 which are the eigenvalues of the matrix. The vibration modes associated with each frequency will then be the eigenvectors.

It should also be noted that Equation B.5 can be written in the form

$$\begin{bmatrix} M^{-1} K - \omega^2 I \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = 0$$

EXAMPLE B.1 - TWO DEGREE OF FREEDOM SYSTEM

Determine the eigenvalues (frequencies) and eigenvectors (modes) for the two degree of freedom mass-spring system shown in Figure B.2.

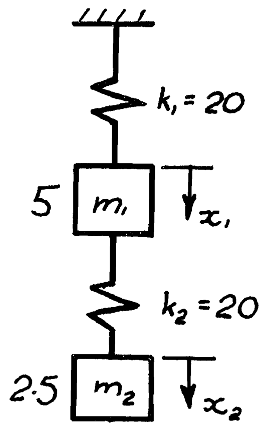


FIGURE B.2

SOLUTION

The problem will be solved in a step by step procedure for clarity.

(a) Stiffness Matrix

The Stiffness Matrix is formed in a similar manner to that previously described, i.e. the Primary Stiffness Matrix is first formed and then reduced to the Structure Stiffness Matrix by invoking constraint conditions. Hence

$$K = \begin{bmatrix} 40 & -20 \\ -20 & 20 \end{bmatrix}$$

SOLUTION (Contd.)

OR

$$\begin{bmatrix} P_1 \\ \hline P_2 \end{bmatrix} = \left[\begin{array}{c|c} 40 & -20 \\ \hline -20 & 20 \end{array}\right] \begin{bmatrix} x_1 \\ \hline x_2 \end{bmatrix}$$

(b) Mass Matrix

This is simply a diagonal matrix of the masses taken in turn.

$$M = \left[\begin{array}{c|c} 5 & 0 \\ \hline 0 & 2.5 \end{array}\right]$$

(c) Equations of Motion

The two equations of motion are given by the relationship

$$K\,X = \omega^2\,M\,X$$

OR

$$\begin{bmatrix} 40 & -20 \\ -20 & 20 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \omega^2 \begin{bmatrix} 5 & 0 \\ 0 & 2.5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Taking the right hand side of the above system of equations to the left hand side and subtracting results in

$$\begin{bmatrix} 40 - 5\omega^2 & -20 \\ -20 & 20 - 2.5\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \quad \text{-----(B.6)}$$

(d) Eigenvalues

Expanding the determinant of the matrix of Equation B.6 to obtain the non-trivial solutions gives

$$(40 - 5\omega^2)\ (20 - 2.5\omega^2) - (20.20) = 0 \quad \text{------ (a)}$$

$$800 - 200\omega^2 + 12.5\omega^4 - 400 = 0$$

$$12.5\omega^4 - 200\omega^2 + 400 = 0 \quad \text{------ (b)}$$

Dividing Equation (b) thru by 12.5

$$\omega^4 - 16\omega^2 + 32 = 0$$

which is a quadratic equation in ω^2. The roots of this equation, i.e. the eigenvalues, become

$$\omega_1^2 = 4(2 - \sqrt{2})$$

$$\omega_2^2 = 4(2 + \sqrt{2})$$

SOLUTION (Contd.)

(e) Eigenvectors

Expanding the first row of Equation 2.20 thus

$$(40 - 5\omega^2)\ X_1 \ - \ 20\ X_2 \quad = \quad 0$$

$$X_2 = \frac{40 - 5\omega^2}{20}\ X_1 = \frac{8 - \omega^2}{4}\ X_1 \qquad \text{--------- (c)}$$

Substituting for ω^2 in Equation (c)

$$X_2 = \frac{8 - 8 + 4\sqrt{2}}{4}\ X_1$$

$$X_2 = \sqrt{2}\ X_1$$

Hence

$$_1X = \begin{bmatrix} _1X_1 \\ _1X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

which is the first eigenvector or mode of vibration of the system.

When

$$\omega^2 = 4\ (2 + \sqrt{2}\ )$$

then

$$X_2 = \frac{8 - 8 - 4\sqrt{2}}{4}\ X_1$$

$$X_2 = -\sqrt{2}\ X_1$$

hence

$$_2X = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}$$

the second eigenvector or mode associated with the higher eigenvalue or frequency.

ALTERNATIVE FORMULATION OF THE EIGEN-PROBLEM

It has already been shown that

$$KX = \omega^2\ MX \qquad \text{----------(B.7)}$$

Premultiplying both sides of Equation B.7 by K^{-1} results in

$$K^{-1}\ MX = \frac{1}{\omega^2}\ X$$

ALTERNATIVE FORMULATION OF THE EIGEN-PROBLEM (Contd.)

but

$$K^{-1} = F = \text{Flexibility Matrix.}$$

Hence the alternative form of the Equations of Motion become

$$\boxed{FMX = \frac{1}{\omega^2} X} \qquad \text{(B.8)}$$

In a number of cases it is not convenient to form the stiffness matrix and an alternative flexibility approach is favoured.

EXAMPLE B.2 - BEAM VIBRATIONS

Determine the frequencies and vibration modes for the simply supported concrete beam shown in Figure B.3. E for the material is 2.5×10^7 kN/m^2, I for the section is 0.06 m^4 and $m_1 = m_2 = 0.7$ Mg.

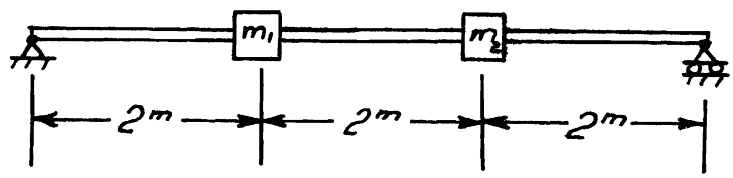


FIGURE B.3

SOLUTION

A step by step procedure of solution will once again by used. A table of influence coefficients has been used to obtain the coefficients of the flexibility matrix.

(a) Flexibility Matrix

Figure B.4 shows the beam with its associated displacement coordinates and loads.

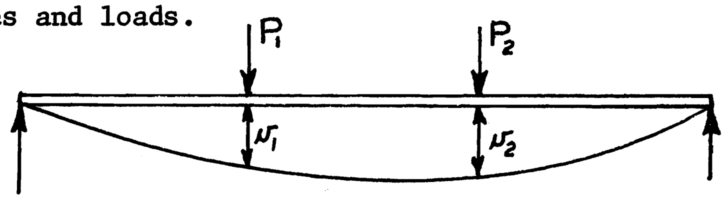


FIGURE B.4

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 3.57 & 3.14 \\ 3.14 & 3.57 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

SOLUTION (Contd.)

(b) Equations of Motion

In this case the two Equations of motion are given by

$$FMX = \frac{1}{\omega^2} X$$

which on substitution yields

$$\frac{1}{EI}\begin{bmatrix} 3.57 & 3.14 \\ 3.14 & 3.57 \end{bmatrix} \begin{bmatrix} 0.7 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{\omega^2} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Writing

$$\lambda = \frac{EI}{\omega^2}$$

and expanding the above matrix relationship

$$\begin{bmatrix} 2.5 - \lambda & 2.2 \\ 2.2 & 2.5 - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \text{------(B.9)}$$

(c) Eigenvalues

Expanding the determinant of the matrix of Equation B.9 gives

$$(2.5 - \lambda)\ (2.5 - \lambda) - (2.2)^2 = 0$$

$$6.25 - 5\lambda + \lambda^2 - 4.48 = 0$$

$$\lambda^2 - 5\lambda + 1.41 = 0$$

which is a quadratic in λ. The roots of this equation are

$$\lambda_1 = 4.7$$

$$\lambda_2 = 0.3$$

from which

$$\omega_1^2 = \frac{EI}{4.7}$$

$$\omega_2^2 = \frac{EI}{0.3}$$

hence

$$\omega_1 = \sqrt{\frac{0.06 \times 2.5 \times 10^7}{4.7}} = 800 \text{ rad/sec.}$$

$$\omega_2 = \sqrt{\frac{0.06 \times 2.5 \times 10^7}{0.3}} = 12{,}560 \text{ rad/sec.}$$

SOLUTION (Contd.)

(d) Eigenvectors

Taking the first root i.e. $\lambda_1 = 4.7$ and expanding the first row of Equation 2.23

$$-2.2v_1 + 2.2v_2 = 0$$

$$1V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 0.3$

$$2.2v_1 + 2.2v_2 = 0$$

$$V_1 = -V_2$$

$$2V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

MODES OF VIBRATION

Figure B.5 illustrates diagrammatically the two modes of vibration of the beam.

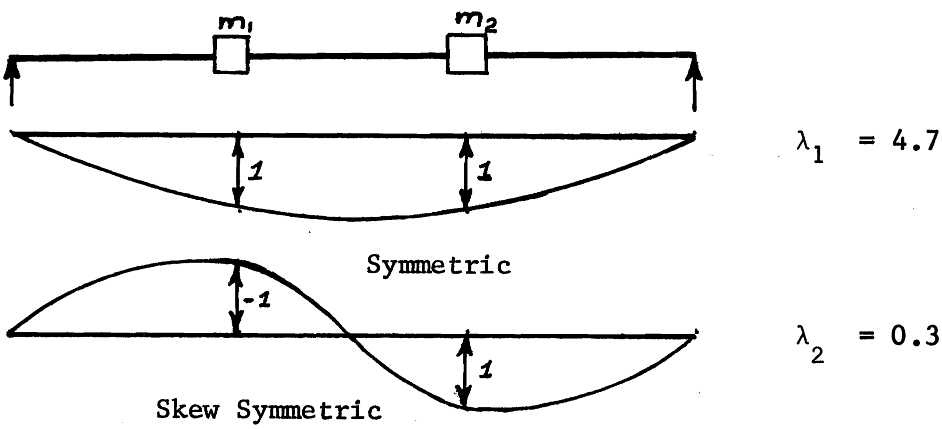


FIGURE B.5

CLOSURE TO STRUCTURAL DYNAMICS

The basics of matrix structural dynamics have been presented in this Appendix in that the single case of free, undamped vibrations has been considered.

It should be noted that the foregoing presentation produces the natural frequencies of vibration of the system together with the relative displacements of the masses. To obtain the actual displacements of the masses at some time t however, requires further work which is not considered warranted at this stage.

CONCLUSION

Only the Direct Stiffness Method has been considered in this book mainly because it lends itself so amicably to automatic computational techniques. It should be constantly borne in mind by the student that to extract the most from the application of the Stiffness Method, access to a computer with reasonable core store is absolutely essential. The Stiffness Method, other than for the most trivial problems, is not a recommended hand method of solution. The method comes into its own for obtaining solutions to problems with many degrees of freedom. It will no doubt have been observed that the word indeterminacy did not have to be mentioned during formulation. This factor in itself is highly significant when one is intent on understanding the fundamental structural concepts of equilibrium and compatibility.

The contents of this book are considered by the author to be a minimum coverage of undergraduate requirements for matrix methods of structural analysis. The work presented in the book constitutes the basis of a 2 hour/week, one semester course for all 3rd year Civil Engineering students at C.I.A.E. It is felt that should the student choose to discontinue his studies of structures in the 4th year of the Civil Course he should have at least sufficient basic knowledge of modern methods of structural analysis to undergo a self educating programme on graduation.

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