Delay-Dependent Robust H_∞ Filtering for Uncertain Discrete-Time Systems With Time-Varying Delay Based on a Finite Sum Inequality

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***Abstract*—This brief is concerned with delay-dependent robust H_∞ filtering for uncertain discrete-time systems with time-varying delay. The uncertainty is of convex polytopic type. By establishing a finite sum inequality based on quadratic terms, a new delay-dependent bounded real lemma (BRL) is derived. In combination with a parameter-dependent Lyapunov–Krasovskii functional, which allows the Lyapunov–Krasovskii matrices to be vertex dependent, the obtained BRL is modified into a new version to suit for convex polytopic uncertainties. Neither model transformation nor bounding technique for cross terms is involved. Based on the new BRL, the designed filter is provided in terms of a linear matrix inequality (LMI), which is easily solved by Matlab LMI toolbox. A numerical example is given to illustrate the effectiveness of the proposed method.**

***Index Terms*—Discrete-time linear systems, H_∞ filtering, linear matrix inequality (LMI), stability, time-varying delay.**

I. Introduction

The topic of H_∞ filtering for systems has been widely investigated in the past decade (see, for example, [2], [5], [6], [9], [12], and references therein). The aim is to design a suitable and stable filter to minimize the H_∞ norm such that the induced $\mathcal{L}_2/l_2$ gain from the noise signal to the estimation error is less than a prescribed level. Some approaches to this issue, such as the algebraic Riccati equations/inequalities, interpolation, and linear matrix inequalities (LMIs), have been developed in the recent years.

A great number of practical systems are unavoidably subject to uncertainties and delays, which usually degrade the performance of the systems under consideration. Therefore, the robust H_∞ filtering for uncertain systems with time delay have been focused on recently. For discrete-time systems with time delay, some sufficient conditions were obtained for the existence of the desired filters [3], [4], [8], [10]. However, these conditions are delay independent, which are conservative, especially for small delay. In [1], a model transformation approach incorporating with Moon's inequality to bound some cross terms was employed, and some delay-dependent conditions were derived to design the desired H_∞ filter. As is well known, these conditions may be also conservative due to the model transformation and bounding technique for cross terms. Moreover, the aforementioned results were established under the assumption that the delay was constant; when the delay is time varying, they are inapplicable. To the best of our knowledge, no delay-dependent filtering result for discrete-time systems with time-varying delay has been reported in the open literature.

In this brief, we will design the delay-dependent H_∞ filter for discrete-time systems with time-varying delay. A new sufficient criterion for the existence of a suitable H_∞ filter will be obtained by introducing a new finite sum inequality, which avoids using both model transformation and bounding technique for cross terms. A numerical example will be given to show that the obtained results are less conservative than some existing ones.

Notation: The symmetric term in a symmetric matrix is denoted by $*$, e.g., $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$. The other notations are routine ones.

II. Problem Statement

Consider the following uncertain discrete-time system with time-varying delay:

$$\begin{cases} x(k+1) = A_0x(k) + A_1x\left(k-h(k)\right) + B_1w(k) \\ y(k) = C_0x(k) + C_1x\left(k-h(k)\right) + B_2w(k) \\ z(k) = L_0x(k) + L_1x\left(k-h(k)\right) + B_3w(k) \\ x(k) = \phi(k), \qquad k = -\bar{h}, -\bar{h}+1, \ldots, 0 \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^m$ is the measured output, $z(k) \in \mathbb{R}^p$ is the signal to be estimated, $w(k) \in \mathbb{R}^q$ is assumed to be an arbitrary noise signal in l_2, $\{\phi(k),\ k = -\bar{h}, -\bar{h}+1, \ldots, 0\}$ is a known given initial condition sequence, and $h(k)$ is a positive integer time-varying delay satisfying

$$0 < \underline{h} \le h(k) \le \bar{h} < \infty, \qquad k = 1, 2, \ldots. \tag{2}$$

Denote $d := \bar{h} - \underline{h}$. Clearly, $d = 0$ means that the time-delay $h(k)$ is time invariant.

The system matrices are supposed to be uncertain and unknown but belong to a known convex compact set of polytopic type, i.e.,

$$\chi := (A_0, A_1, B_1, C_0, C_1, B_2, L_0, L_1, B_3) \in \Omega \tag{3}$$

Manuscript received March 21, 2006; revised May 30, 2006. This work was supported in part by Central Queensland University for the Research Advancement Awards Scheme Project "Robust Fault Detection, Filtering and Control for Uncertain Systems With Time-Varying Delay" (January 2006–December 2008). This paper was recommended by Associate Editor Y. Lian.

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Digital Object Identifier 10.1109/TCSII.2006.884116

where

$$\Omega := \left\{ \chi | \chi = \sum_{i=1}^{r} \rho_i \chi_i, \quad \sum_{i=1}^{r} \rho_i = 1, \quad \rho_i \geq 0 \right\}$$

with $\chi_i := (A_{0i}, A_{1i}, B_{1i}, C_{0i}, C_{1i}, B_{2i}, L_{0i}, L_{1i}, B_{3i})$, which denotes the ith vertex of the polyhedral domain Ω.

Suppose that system (1) is robustly asymptotically stable over the entire polytopic domain Ω. In this case, we will design a full-order filter with state-space realization of the following form:

$$\begin{cases} \hat{x}(k+1) = A_f \hat{x}(k) + B_f y(k), & \hat{x}(0) = 0 \\ \hat{z}(k) = C_f \hat{x}(k) + D_f y(k) \end{cases} \tag{4}$$

where constant matrices $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times m}$, $C_f \in \mathbb{R}^{p \times n}$, and $D_f \in \mathbb{R}^{p \times m}$ are filter parameters to be determined.

Defining the augmented state vector $\tilde{x}(k) := [x^T(k), \hat{x}^T(k)]^T$ and the estimation error $\tilde{z}(k) := z(k) - \hat{z}(k)$, one obtains the following filtering system:

$$\begin{cases} \tilde{x}(k+1) = \tilde{A}_0 \tilde{x}(k) + \tilde{A}_1 E \tilde{x}(k - h(k)) + \tilde{B}_1 w(k) \\ \tilde{z}(k) = \tilde{L}_0 \tilde{x}(k) + \tilde{L}_1 E \tilde{x}(k - h(k)) + \tilde{B}_3 w(k) \\ \tilde{x}(k) = \left[\phi^T(k), 0\right]^T, \qquad k = -\bar{h}, -\bar{h}+1, \ldots, 0 \end{cases} \tag{5}$$

where $E = [I \quad 0]$ and

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_f C_0 & A_f \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A_1 \\ B_f C_1 \end{bmatrix}$$
$$\tilde{B}_1 = \begin{bmatrix} B_1 \\ B_f B_2 \end{bmatrix}, \quad \tilde{L}_0 = [\, L_0 - D_f C_0 \quad - C_f \,]$$
$$\tilde{L}_1 = L_1 - D_f C_1, \quad \tilde{B}_3 = B_3 - D_f B_2.$$

The purpose of this brief is to design a robust filter of the form (4) such that the system (5) has a prescribed H_∞ performance for all uncertainties satisfying (3).

1) System (5) with $w(k) = 0$ is asymptotically stable.
2) System (5) has a prescribed level γ of H_∞ noise attenuation, i.e., under the zero initial condition, $\|\tilde{z}\|_2 < \gamma \|w\|_2$ is satisfied for any nonzero $w \in l_2$.

In the following, we introduce two vectors:

$$\xi(k) := [\, \tilde{x}^T(k) \quad \tilde{x}^T(k - h(k))\, E^T \quad w^T(k) \,]^T$$
$$\tilde{y}(k) := \tilde{x}(k+1) - \tilde{x}(k)$$

then

$$\tilde{x}(k+1) = \Gamma_1 \xi(k), \quad E\tilde{y}(k) = \Gamma_2 \xi(k), \quad \tilde{z}(k) = \Gamma_3 \xi(k) \tag{6}$$

where

$$\Gamma_1 := [\, \tilde{A}_0 \quad \tilde{A}_1 \quad \tilde{B}_1 \,]$$
$$\Gamma_2 := [\, (A_0 - I)E \quad A_1 \quad B_1 \,]$$
$$\Gamma_3 := [\, \tilde{L}_0 \quad \tilde{L}_1 \quad \tilde{B}_3 \,].$$

The following lemma gives the relationship between the vectors $\xi(k)$ and $E\tilde{y}(k)$, which will play a key role in the delay-dependent H_∞ performance analysis.

Lemma 1: For any constant matrices $M := [M_1 \quad M_2] \in \mathbb{R}^{n \times 2n}$, $M_3 \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{n \times q}$, $R \in \mathbb{R}^{n \times n}$ with $R > 0$, and a positive integer time-varying $h(k)$, then

$$- \sum_{j=k-h(k)}^{k-1} \tilde{y}^T(j) E^T R E \tilde{y}(j) \leq \xi^T(k) \left\{ \beth + h(k) Y^T R^{-1} Y \right\} \xi(k) \tag{7}$$

where

$$\beth := \begin{bmatrix} M^T E + E^T M & -M^T + E^T M_3 & E^T W \\ * & -M_3^T - M_3 & -W \\ * & * & 0 \end{bmatrix} \tag{8}$$
$$Y := [\, M \quad M_3 \quad W \,]. \tag{9}$$

Proof: Let $\mathcal{C} = \begin{bmatrix} R^{1/2} & R^{-1/2} Y \\ 0 & 0 \end{bmatrix}$, then

$$\begin{bmatrix} R & Y \\ Y^T & Y^T R^{-1} Y \end{bmatrix} = \mathcal{C}^T \mathcal{C} \geq 0.$$

It follows that

$$\sum_{j=k-h(k)}^{k-1} \begin{bmatrix} E\tilde{y}(j) \\ \xi(k) \end{bmatrix}^T \begin{bmatrix} R & Y \\ Y^T & Y^T R^{-1} Y \end{bmatrix} \begin{bmatrix} E\tilde{y}(j) \\ \xi(k) \end{bmatrix} \geq 0. \tag{10}$$

Notice that

$$\sum_{j=k-h(k)}^{k-1} 2\xi^T(k) Y^T E \tilde{y}(k) = 2\xi^T(k) Y^T [E \quad -I \quad 0] \xi(k).$$

Rearranging (10) yields (7). ■

III. H_∞ Performance Analysis

Based on Lemma 1, a new delay-dependent condition on H_∞ performance analysis is derived, which can guarantee that the filtering system (5) has a prescribed H_∞ performance γ.

Proposition 1: Given $\gamma > 0$, the system (5) with $\chi \in \Omega$ is asymptotically stable with a guaranteed H_∞ performance γ if there exist real matrices $P > 0$, $R > 0$, $Q > 0$, $M := [M_1 \quad M_2]$, M_3, and W with appropriate dimensions such that

$$\begin{bmatrix} \Xi & \Pi \\ * & \Lambda \end{bmatrix} < 0 \tag{11}$$

where

$$\Xi := \begin{bmatrix} \Xi_{11} & \Xi_{12} & E^T W \\ * & \Xi_{22} & -W \\ * & * & -\gamma^2 I \end{bmatrix}$$
$$\Pi := \begin{bmatrix} \bar{h} M^T & \bar{h} E^T (A_0 - I)^T & \tilde{L}_0^T & \tilde{A}_0^T \\ \bar{h} M_3^T & \bar{h} A_1^T & \tilde{L}_1^T & \tilde{A}_1^T \\ \bar{h} W^T & \bar{h} B_1^T & \tilde{B}_3^T & \tilde{B}_1^T \end{bmatrix}$$
$$\Lambda := \mathrm{diag}\{-\bar{h} R, -\bar{h} R^{-1}, -I, -P^{-1}\}$$
$$\Xi_{11} := M^T E + E^T M - P + (d+1) E^T Q E$$
$$\Xi_{12} := -M^T + E^T M_3$$
$$\Xi_{22} := -M_3^T - M_3 - Q.$$

Proof: Choose a Lyapunov–Krasovskii functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) \tag{12}$$

where

$$\begin{aligned} V_1(k) &= \tilde{x}^T(k) P \tilde{x}(k) \\ V_2(k) &= \sum_{\theta=-\bar{h}+1}^{0} \sum_{j=k-1+\theta}^{k-1} \tilde{y}^T(j) E^T R E \tilde{y}(j) \\ V_3(k) &= \sum_{i=k-h(k)}^{k-1} \tilde{x}^T(i) E^T Q E \tilde{x}(i) \\ V_4(k) &= \sum_{j=-\bar{h}+2}^{-\underline{h}+1} \sum_{l=k+j-1}^{k-1} \tilde{x}^T(l) E^T Q E \tilde{x}(l) \end{aligned}$$

with $P > 0$, $R > 0$, and $Q > 0$ are to be determined.

Taking the forward difference $\Delta V(k) = V(k+1) - V(k)$ along the trajectory of system (5) yields

$$\begin{aligned} \Delta V_1(k) &= \tilde{x}^T(k+1) P \tilde{x}(k+1) - \tilde{x}^T(k) P \tilde{x}(k) \\ &= \xi^T(k) \Gamma_1^T P \Gamma_1 \xi(k) - \tilde{x}^T(k) P \tilde{x}(k) && (13) \\ \Delta V_2(k) &= \bar{h} \xi^T(k) \Gamma_2^T R \Gamma_2 \xi(k) \\ &\quad - \sum_{j=k-\bar{h}}^{k-1} \tilde{y}^T(j) E^T R E \tilde{y}(j). && (14) \end{aligned}$$

Noting that $h(k) \leq \bar{h}$ from (2), we derive

$$-\sum_{j=k-\bar{h}}^{k-1} y^T(j) R y(j) \leq -\sum_{j=k-h(k)}^{k-1} y^T(j) R y(j). \tag{15}$$

Use Lemma 1 to obtain

$$\Delta V_2(k) \leq \xi(k) \left\{ \bar{h} \Gamma_2^T R \Gamma_2 + \bar{h} Y^T R^{-1} Y + \daleth \right\} \xi(k) \tag{16}$$

where $\daleth$ and Y are defined in (8) and (9), respectively. Similar to [11], we have

$$\begin{aligned} \Delta V_3(k) + \Delta V_4(k) &\leq (\bar{h} - \underline{h} + 1) \tilde{x}^T(k) E^T Q E \tilde{x}(k) \\ &\quad - \tilde{x}^T(k - h(k)) E^T Q E \tilde{x}(k - h(k)). && (17) \end{aligned}$$

Combining (13), (16), and (17), one obtains

$$\begin{aligned} \Delta V(k) - \gamma^2 w^T(k) w(k) &\leq \xi^T(k) \\ &\times \left\{ \Xi + \Gamma_1^T P \Gamma_1 + \bar{h} \Gamma_2^T R \Gamma_2 + \bar{h} Y^T R^{-1} Y \right\} \xi(k) && (18) \end{aligned}$$

where Ξ is defined in (11).

We first show that the system (5) with $w(k) = 0$ is asymptotically stable. In fact, (11) implies

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \bar{h} M^T & \bar{h} E^T (A_0 - I)^T & \tilde{A}_0^T \\ * & \Xi_{22} & \bar{h} M_3^T & \bar{h} A_1^T & \tilde{A}_1^T \\ * & * & -\bar{h} R & 0 & 0 \\ * & * & * & -\bar{h} R^{-1} & 0 \\ * & * & * & * & -P^{-1} \end{bmatrix} < 0. \tag{19}$$

Using Schur complement and considering (18) with $w(k) = 0$ yields $\Delta V(k) < 0$, which guarantees the asymptotic stability of (5) with $w(k) = 0$.

Next, assuming that under zero initial condition, $\|\tilde{z}\|_2 < \gamma \|w\|_2$ is satisfied for any nonzero $w \in l_2$. Noting that $z^T(k) z(k) = \xi^T(k) \Gamma_3^T \Gamma_3 \xi(k)$, rewrite (18) as

$$\begin{aligned} \Delta V(k) - \gamma^2 w^T(k) w(k) + \tilde{z}^T(k) \tilde{z}(k) &\leq \xi^T(k) \\ \times \left\{ \Xi + \Gamma_1^T P \Gamma_1 + \bar{h} \Gamma_2^T R \Gamma_2 + \Gamma_3^T \Gamma_3 + \bar{h} Y^T R^{-1} Y \right\} \xi(k). \end{aligned}$$

Thus, if (11) holds, using Schur complement yields

$$\Delta V(k) < \gamma^2 w^T(k) w(k) - \tilde{z}^T(k) \tilde{z}(k). \tag{20}$$

Summing both sides of (20) from 0 to ∞, we obtain

$$\sum_{k=0}^{\infty} \tilde{z}^T(k) \tilde{z}(k) < \sum_{k=0}^{\infty} \gamma^2 w^T(k) w(k) + V(0) - V(\infty). \tag{21}$$

Under zero initial condition, $V(0) = 0$, one obtains

$$\sum_{k=0}^{\infty} \tilde{z}^T(k) \tilde{z}(k) < \sum_{k=0}^{\infty} \gamma^2 w^T(k) w(k). \tag{22}$$

That is, $\|\tilde{z}\|_2 < \gamma \|w\|_2$ is true for all nonzero $w(k) \in l_2$, which completes the proof. ■

Remark 1: From the proof process of Proposition 1, one can clearly see that neither model transformation nor bounding technique for cross terms is involved. Therefore, the obtained result is expected to be less conservative.

To exploit parameter-dependent Lyapunov–Krasovskii functionals to handle the polytopic uncertainties, using the idea in [7] by introducing two slack variables H_1 and H_2, we can conclude that the matrix (11) is implied by

$$\begin{bmatrix} \Xi & \tilde{\Pi} \\ * & \tilde{\Lambda} \end{bmatrix} < 0 \tag{23}$$

where

$$\begin{aligned} \tilde{\Pi} &:= \begin{bmatrix} \bar{h} M^T & \bar{h} E^T (A_0 - I)^T H_1^T & \tilde{L}_0^T & \tilde{A}_0^T H_2^T \\ \bar{h} M_3^T & \bar{h} A_1^T H_1^T & \tilde{L}_1^T & \tilde{A}_1^T H_2^T \\ \bar{h} W^T & \bar{h} B_1^T H_1^T & \tilde{B}_3^T & \tilde{B}_1^T H_2^T \end{bmatrix} \\ \tilde{\Lambda} &:= \mathrm{diag} \left\{ -\bar{h} R, \bar{h} \left(R - H_1^T - H_1 \right), -I, P - H_2^T - H_2 \right\}. \end{aligned}$$

Then, employing parameter-dependent Lyapunov–Krasovskii functionals yields the following conclusion.

Proposition 2: Given $\gamma > 0$, the filtering system (5) with (2) is asymptotically stable with a guaranteed γ level of noise attenuation for all uncertainties satisfying (3) if there exist real $n \times n$ matrices $P_i > 0$, $R_i > 0$, $Q_i > 0$, M_{ji} $(j = 1, 2, 3)$, H_1, H_2, and $n \times q$ matrices W_i such that for $i = 1, 2, \ldots, r$

$$\Psi_{1i} := \begin{bmatrix} \Xi_i & \tilde{\Pi}_i \\ * & \tilde{\Lambda}_i \end{bmatrix} < 0 \tag{24}$$

where

$$\Xi_i := \begin{bmatrix} \Xi_{11i} & \Xi_{12i} & E^T W_i \\ * & \Xi_{22i} & -W_i \\ * & * & -\gamma^2 I \end{bmatrix}$$
$$\tilde{\Pi}_i := \begin{bmatrix} \bar{h} M_i^T & \bar{h} E^T (A_{0i} - I)^T H_1^T & \tilde{L}_{0i}^T & \tilde{A}_{0i}^T H_2^T \\ \bar{h} M_{3i}^T & \bar{h} A_{1i}^T H_1^T & \tilde{L}_{1i}^T & \tilde{A}_{1i}^T H_2^T \\ \bar{h} W_i^T & \bar{h} B_{1i}^T H_1^T & \tilde{B}_{3i}^T & \tilde{B}_{1i}^T H_2^T \end{bmatrix}$$
$$\tilde{\Lambda}_i := \mathrm{diag}\left\{-\bar{h} R_i, \bar{h}\left(R_i - H_1^T - H_1\right), -I, P_i - H_2^T - H_2\right\}$$
$$\Xi_{11i} := M_i^T E + E^T M_i - P_i + (d+1) E^T Q_i E$$
$$\Xi_{12i} := -M_i^T + E^T M_{3i}$$
$$\Xi_{22i} := -M_{3i}^T - M_{3i} - Q_i$$
$$M_i := [M_{1i} \quad M_{2i}].$$

IV. H_∞ Filter Design

In this section, based on Proposition 2, we present the following sufficient condition for the existence of a desired filter of form (4).

Proposition 3: Consider the system (1) with uncertainties (3), an admissible robust H_∞ filter of the form (4) exists if there exist matrix $\begin{bmatrix} P_{1i} & P_{2i} \\ * & P_{3i} \end{bmatrix} > 0$ with $P_{1i}, P_{2i}, P_{3i} \in \mathbb{R}^{n\times n}$, $R_i > 0$, $Q_i > 0$, $M_{1i}, \tilde{M}_{2i}, M_{3i}, H_1, \mathcal{F}_1, \mathcal{F}_2, \mathcal{A} \in \mathbb{R}^{n\times n}$, and matrices $W_i \in \mathbb{R}^{n\times q}, \mathcal{B} \in \mathbb{R}^{n\times m}, \mathcal{C} \in \mathbb{R}^{p\times n}$, and $D_f \in \mathbb{R}^{p\times m}$ such that for $i = 1, 2, \ldots, r$

$$\Psi_{2i} := \begin{bmatrix} \Theta_{1i} & \Theta_{2i} & \Theta_{3i} \\ * & \Theta_{4i} & 0 \\ * & * & \Theta_{5i} \end{bmatrix} < 0 \tag{25}$$

where

$$\Theta_{1i} := \begin{bmatrix} \varphi_i & \tilde{M}_{2i} - P_{2i} & -M_{1i}^T + M_{3i} & W_i \\ * & -P_{3i} & -\tilde{M}_{2i}^T & 0 \\ * & * & -M_{3i} - M_{3i}^T - Q_i & -W_i \\ * & * & * & -\gamma^2 I \end{bmatrix}$$
$$\Theta_{2i} := \begin{bmatrix} \bar{h} M_{1i}^T & \bar{h}(A_{0i} - I)^T H_1^T & L_{0i}^T - C_{0i}^T D_f^T \\ \bar{h} \tilde{M}_{2i}^T & 0 & -\mathcal{C}^T \\ \bar{h} M_{3i}^T & \bar{h} A_{1i}^T H_1^T & L_{1i}^T - C_{1i}^T D_f^T \\ \bar{h} W_i^T & \bar{h} B_{1i}^T H_1^T & B_{3i}^T - B_{2i}^T D_f^T \end{bmatrix}$$
$$\Theta_{3i} := \begin{bmatrix} A_{0i}^T X_1^T + C_{0i}^T \mathcal{B}^T & A_{0i}^T \mathcal{F}_1^T + C_{0i}^T \mathcal{B}^T \\ \mathcal{A}^T & \mathcal{A}^T \\ A_{1i}^T X_1^T + C_{1i}^T \mathcal{B}^T & A_{1i}^T \mathcal{F}_1^T + C_{1i}^T \mathcal{B}^T \\ B_{1i}^T X_1^T + B_{2i}^T \mathcal{B}^T & B_{1i}^T \mathcal{F}_1^T + B_{2i}^T \mathcal{B}^T \end{bmatrix}$$
$$\Theta_{4i} := \mathrm{diag}\left\{-\bar{h} R_i, \bar{h}\left(R_i - H_1^T - H_1\right), -I\right\}$$
$$\Theta_{5i} := \begin{bmatrix} P_{1i} - X_1^T - X_1 & P_{2i} - \mathcal{F}_1^T - \mathcal{F}_2 \\ * & P_{3i} - \mathcal{F}_2^T - \mathcal{F}_2 \end{bmatrix}$$
$$\varphi_i := M_{1i}^T + M_{1i} - P_{1i} + (d+1) Q_i.$$

Moreover, a suitable filter realization is given by

$$A_f = \mathcal{A}\mathcal{F}_2^{-1}, \quad B_f = \mathcal{B}, \quad C_f = \mathcal{C}\mathcal{F}_2^{-1}, \quad D_f = D_f. \tag{26}$$

Proof: We are about to prove the conclusion using Proposition 2. If the matrix inequality (24) is true, then $H_2 + H_2^T > 0$. Partition H_2 as

$$H_2 = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

where $X_1, X_4 \in \mathbb{R}^{n\times n}$. Then, $X_4 + X_4^T > 0$, from which we can deduce that X_4 is invertible. Define

$$J := \begin{bmatrix} I & 0 \\ 0 & X_2 X_4^{-1} \end{bmatrix}, \quad \mathcal{F}_1 := X_2 X_4^{-1} X_3$$
$$\mathcal{F}_2 := X_2 X_4^{-T} X_2^T, \quad \tilde{M}_{2i} := M_{2i} X_4^{-T} X_2^T$$
$$\mathcal{T} := \mathrm{diag}\{J, I, I, I, I, I, J\}, \quad \begin{bmatrix} P_{1i} & P_{2i} \\ P_{2i}^T & P_{3i} \end{bmatrix} := J P_i J^T$$

and let

$$\begin{cases} \mathcal{A} := X_2 A_f X_4^{-T} X_2^T \\ \mathcal{B} := X_2 B_f \\ \mathcal{C} := C_f X_4^{-T} X_2^T. \end{cases} \tag{27}$$

Then, we have

$$\mathcal{T} \Psi_{1i} \mathcal{T}^T = \Psi_{2i}, \qquad i = 1, 2 \ldots, r \tag{28}$$

where Ψ_{1i} and Ψ_{2i} are defined in (24) and (25), respectively. If (25) holds, i.e., $\Psi_{2i} < 0$, then $\Psi_{1i} < 0$ is true. Therefore, the filtering system (5) has a prescribed H_∞ performance γ.

On the other hand, clearly, the filter parameters A_f, B_f, C_f, and D_f are implicit in (27) except that D_f can be directly obtained from (25). To solve the other filter parameters A_f, B_f, and C_f, at first, it is obvious that X_2 is invertible from (28) if (25) is feasible, then the filter parameters A_f, B_f, and C_f can be rewritten as

$$\begin{cases} A_f = X_2^{-1} \mathcal{A} \mathcal{F}_2^{-1} X_2 \\ B_f = X_2^{-1} \mathcal{B} \\ C_f = \mathcal{C} \mathcal{F}_2^{-1} X_2^{-1}. \end{cases} \tag{29}$$

Since the following systems are algebraically equivalent:

$$\left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right] = \left[\begin{array}{c|c} X_2^{-1} \mathcal{A} \mathcal{F}_2^{-1} X_2 & X_2^{-1} \mathcal{B} \\ \hline \mathcal{C} \mathcal{F}_2^{-1} X_2^{-1} & D_f \end{array}\right] \Longleftrightarrow \left[\begin{array}{c|c} \mathcal{A} \mathcal{F}_2^{-1} & \mathcal{B} \\ \hline \mathcal{C} \mathcal{F}_2^{-1} & D_f \end{array}\right] \tag{30}$$

thus a state-space realization as (26) of the desired filter is readily obtained from (30), which completes the proof. ■

Remark 2: Proposition 3 provides a delay-dependent condition to design a suitable H_∞ filter for uncertain discrete-time systems with time-varying delay. This condition depends on the upper bound as well as the lower bound of the time-varying delay. Thus, when these bounds are available, Proposition 3 can achieve less conservative results. Moreover, if the lower bound

TABLE I
ACHIEVED MINIMUM H_∞ LEVEL $\gamma_{\min}$ FOR DIFFERENT $\underline{h}$ WHEN $\bar{h} = 5$

$\underline{h}$	1	2	3	4	5
$\gamma_{\min}$	7.1709	5.4786	4.4587	3.7035	3.3957

TABLE II
ACHIEVED MINIMUM H_∞ LEVEL $\gamma_{\min}$ OF H_∞ NOISE ATTENUATION CORRESPONDING TO DELAY UPPER BOUNDS $\bar{h}$

$\bar{h}$	1	2	3	4	5	6	7	8
[4]	2.32	2.49	2.85	3.43	4.31	5.61	8.55	∞
Prop 3	2.30	2.39	2.59	2.93	3.40	4.06	5.37	8.25

is not exactly known, we can replace it with zero; if the lower bound is equal to the upper bound, then the delay is time invariant. In this sense, Proposition 3 can handle the H_∞ filtering for a large class of discrete-time systems.

V. NUMERICAL EXAMPLE

Example : Consider system (1) with

$$\begin{cases} A_0 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.7+\varphi \end{bmatrix}, & A_1 = \begin{bmatrix} -0.1 & \rho \\ -0.1 & -0.1 \end{bmatrix} \\ C_0 = [1 \quad 1], & C_1 = [0.2 \quad 0.5] \\ L_0 = [1 \quad 2], & L_1 = [0.5 \quad 0.6] \\ B_1 = [0 \quad 1]^T, & B_2 = 1, \quad B_3 = -0.5. \end{cases} \tag{32}$$

The uncertain parameters satisfy $|\varphi| \leq 0.2$ and $|\rho| \leq 0.1$.

When $h(k)$ is time varying, the results in [1] are inapplicable to this case. However, using Proposition 3, the achieved H_∞ performances $\gamma_{\min}$ of the filtering system are listed in Table I for different lower bounds $\underline{h}$ when $\bar{h} = 5$. It is clearly shown from this table that $\gamma_{\min}$ decreases as $\underline{h}$ increases.

We now suppose that the delay $h := h(k)$ is time invariant. We calculated the achieved H_∞ performance $\gamma_{\min}$ of filtering system (5) corresponding to different delay upper bounds $\bar{h}$ by using the approaches proposed both in this brief and [1], which are listed in Table II, from which one can clearly see that Proposition 3 obtains much less conservative results than that in [1]. Especially, when $\bar{h} = 8$, Proposition 3 yields the minimum H_∞ level $\gamma_{\min} = 8.25$, and the corresponding filter parameters are given in (33), whereas the method in [1] fails to make any conclusion, i.e.,

$$\left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right] = \left[\begin{array}{cc|c} 1.3703 & 2.6098 & -0.0528 \\ -0.4766 & -0.7762 & 0.0081 \\ \hline -0.9806 & -4.8336 & 1.5093 \end{array}\right]. \tag{33}$$

Connecting filter parameters (33) to the filtering systems (5) and (32), we depicted the singular value curves of the transfer functions at four vertices, which are shown in Fig. 1. Clearly, all of the maximum singular values are less than 8.25, which demonstrates the effectiveness of the proposed method in this brief.

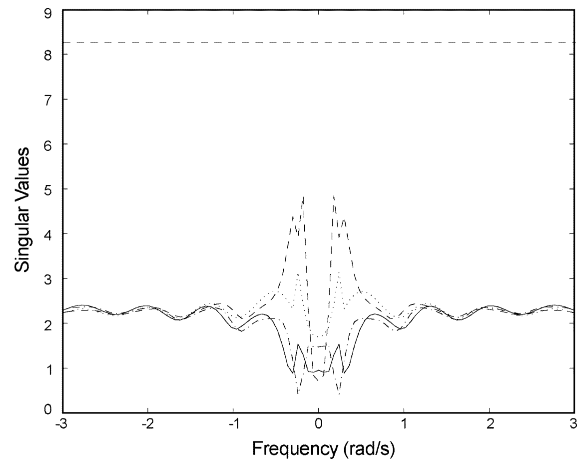

Fig. 1. Maximum singular value curves of the filtering transfer function at the four vertices.

VI. CONCLUSION

The delay-dependent robust H_∞ filtering for uncertain discrete-time systems with time-varying delay has been investigated. By introducing a new finite sum inequality based on quadratic terms, a new bounded real lemma (BRL) for the filtering system has been obtained in combining with Lyapunov–Krasovskii functional method. Neither model transformation nor bounding technique for cross terms has been employed. The new BRL has been modified to be an LMI, which enables us to easily solve the H_∞ filter. Finally, a numerical example have been given to illustrate the effectiveness of the proposed approach.

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