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A New Finite Sum Inequality for Delay-Dependent H_∞ Control of Discrete-Time Delay Systems

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***Abstract*— This paper is concerned with the problem of delay-dependent H_∞ control for linear discrete-time systems with time-varying delay. A new finite sum inequality is first established to derive a delay-dependent condition, under which the resulting closed-loop system is asymptotically stable (internally stable) with a prescribed H_∞ attenuation level via a memoryless state feedback. Then, an iterative algorithm involving convex optimization is proposed to obtain a suboptimal H_∞ controller. Finally, a numerical example is given to show the effectiveness of the proposed method.**

***Index Terms*— Time-varying delay. Discrete-time linear system. H_∞ control. State feedback. Finite sum inequality.**

I. Introduction

Recently, delay-dependent analysis and synthesis of dynamic continuous-time systems with time delay have received considerable attention due to the obtained conditions containing delay size information, and a large number of excellent fruits have been reported in the literature (See [5], [9], [13], [14] and references therein). However, little attention has been paid on the same issues for discrete-time systems with delay. One of the main reasons is that, when the delay is time-invariant, they can be transformed into systems with no time delay via the state augmentation approach [1]. Therefore, the analysis and synthesis problems for discrete-time systems with time-invariant delay can be solved by means of the corresponding theories of discrete-time systems with no delay. Nevertheless, this approach fails to the cases that the delay is time-varying or the system contains uncertainties.

This paper focuses on the delay-dependent H_∞ control of discrete-time systems with both time-varying delay and norm-bounded uncertainties. For this issue, Song *et al.* [10] derived a delay-dependent H_∞ condition based on an LMI, where the H_∞ controller can be obtained by solving an H_∞ control problem for auxiliary discrete-time linear systems with no delay. Fridman and Shaked [4] introduced the descriptor model transformation method to discuss the same issue and some less conservative criteria were obtained by employing Moon et al's inequality to bound two cross-terms. In this paper, different from [10] and [4], a new finite-sum inequality is first introduced to deal with the H_∞ control problem. By employing the finite-sum inequality, a less conservative delay-dependent condition for H_∞ control is obtained. Then, an iterative algorithm involving convex optimization is given to design a suboptimal H_∞ controller, under which the resulting closed-loop system has a prescribed suboptimal H_∞ performance. An example is finally given to illustrate that the proposed method can achieve much less conservative results.

Throughout this paper, $l_2[0,\infty)$ denotes the space of sequences $\{x(k)\}$, $k = 0, 1, 2, \cdots$ with the norm $\|x\|_2^2 = \sum_{k=0}^{\infty} x^T(k)x(k) < \infty$; $P > 0$ means that P is asymptotic positive definite; I is the identity matrix with appropriate dimensions; diag$\{\cdots\}$ denotes a block-diagonal matrix; col$\{\cdots\}$ denotes a column vector; and the symmetric terms in a symmetric matrix are denoted by *, e.g., $\left[\begin{smallmatrix} X & Y \\ * & Z \end{smallmatrix}\right] = \left[\begin{smallmatrix} X & Y \\ Y^T & Z \end{smallmatrix}\right]$.

II. Problem Formulation

Consider the following discrete-time uncertain linear system with time-varying delay:

$$\begin{cases} x(k+1) = (A_0+\Delta A_0(k))x(k)+(A_1+\Delta A_1(k))x(k-d(k)) \\ \qquad\qquad +(B_1+\Delta B_1(k))w(k)+(B_2+\Delta B_2(k))u(k) \\ z(k) = (C_0+\Delta C_0(k))x(k)+(C_1+\Delta C_1(k))x(k-d(k)) \\ \qquad\qquad +(D_{11}+\Delta D_{11}(k))w(k)+(D_{12}+\Delta D_{12}(k))u(k) \\ x(k) = \phi(k),\ -\bar{h} \le k \le 0 \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $z(k) \in \mathbb{R}^p$ are the state, control input and controlled output, respectively. $w(k) \in \mathbb{R}^q$ is the exogenous input, which belongs to $l_2[0,\infty)$. $\phi(k)$ is the initial condition. The coefficient matrices $A_0, A_1, B_1, B_2, C_0, C_1, D_{11}$ and D_{12} are known constant real matrices with appropriate dimensions. The time-varying

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uncertainties considered here are assumed to be of the form

$$\begin{bmatrix} \Delta A_0(k) & \Delta A_1(k) & \Delta B_1(k) & \Delta B_2(k) \\ \Delta C_0(k) & \Delta C_1(k) & \Delta D_{11}(k) & \Delta D_{12}(k) \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} F(k) \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \end{bmatrix} \quad (2)$$

where $F(k)$ is an unknown real time-varying matrix satisfying $F^T(k)F(k) \le I,\ \forall k$; and D_1, D_2 and E_1, E_2, E_3, E_4 are appropriately dimensioned constant matrices that characterize how the uncertainty, $F(k)$, enters the nominal matrices $A_0, A_1, B_1, B_2, C_0, C_1, D_{11}$ and D_{12}. The time-varying delay, $d(k)$, is a positive integer satisfying

$$\underline{h} \le d(k) \le \bar{h},\ \forall k \ge 0. \quad (3)$$

where $\underline{h}$ and $\bar{h}$ are constant positive integers.

Remark 1: Clearly, the time-varying delay $d(k)$ is an interval-like time-varying delay. When $\underline{h} = \bar{h}$ means that the delay $d(k)$ is time-invariant, while $\underline{h} = 1$ stands for

$$0 < d(k) \le \bar{h},\ \forall k \ge 0.$$

which was used in [4], [6], [10].

This paper aims to design a memoryless state feedback controller as

$$u(k) = Kx(k) \quad (4)$$

such that the resulting closed-loop system by (1) and (4) is asymptotically stable with a prescribed H_∞ attenuation level γ, i.e. (i) the closed-loop system is asymptotically stable when $w(k) = 0,\ \forall k > 0$; (ii) the H_∞ performance $\|z\|_2 < \gamma\|w\|_2$, is guaranteed for all nonzero $w(k) \in l_2[0, \infty)$ and and a prescribed $\gamma > 0$ under the condition $\phi(k) = 0,\ -\bar{h} \le k \le 0$, for all uncertainties and time-varying delay satisfying (2) and (3).

In the next sections, two vectors are frequently used:

$$y(k) = x(k+1) - x(k) \quad (5)$$

$$\xi(k) = \mathrm{col}\{x(k), x(k-d(k)), w(k)\} \quad (6)$$

The following lemma discloses the relationship between the vectors $\xi(k)$ and $y(k)$.

Lemma 1: For any matrices $M_1, M_2, Z_{11}, Z_{12}, Z_{22}, R \in \mathbb{R}^{n\times n}$, where $R = R^T$, and $Z_{13}, Z_{23}, M_3 \in \mathbb{R}^{n\times q}$, $Z_{33} \in \mathbb{R}^{q\times q}$, the following inequality holds:

$$-\sum_{j=k-d(k)}^{k-1} y^T(j)Ry(j) \le \xi^T(k) \begin{bmatrix} \upsilon_{11} & \upsilon_{12} & M_3 + \bar{h}Z_{13} \\ * & \upsilon_{22} & -M_3 + \bar{h}Z_{23} \\ * & * & \bar{h}Z_{33} \end{bmatrix} \xi(k) \quad (7)$$

where

$$\begin{bmatrix} R & M_1 & M_2 & M_3 \\ * & Z_{11} & Z_{12} & Z_{13} \\ * & * & Z_{22} & Z_{23} \\ * & * & * & Z_{33} \end{bmatrix} \ge 0 \quad (8)$$

with

$$\upsilon_{11} = M_1^T + M_1 + \bar{h}Z_{11}$$
$$\upsilon_{12} = -M_1^T + M_2 + \bar{h}Z_{12}$$
$$\upsilon_{22} = -M_2^T - M_2 + \bar{h}Z_{22}$$

Proof: Denoting $M = [M_1\ M_2\ M_3]$ and $Z = (Z_{ij})_{3\times 3}$, from (8), we have

$$\sum_{i=k-d(k)}^{k-1} \begin{bmatrix} y(i) \\ \xi(k) \end{bmatrix}^T \begin{bmatrix} R & M \\ * & Z \end{bmatrix} \begin{bmatrix} y(i) \\ \xi(k) \end{bmatrix} \ge 0. \quad (9)$$

After some simple manipulations, (9) gives

$$-\sum_{i=k-d(k)}^{k-1} y^T(i)Ry(i) \le 2\xi^T(k)M^T[I\ \ -I\ \ 0]\xi(k) + d(k)\xi^T(k)Z\xi(k),$$

which gives (7) from (3). □

Remark 2: The formula (7) is called a *finite sum inequality* based on quadratic terms. It plays an important role in the sequence for delay-dependent stability analysis.

III. Main Results

This section presents our main results. First, we consider delay-dependent H_∞ control of the nominal system of (1) with $F(k) = 0, \forall k > 0$. The resulting closed-loop nominal system of (1) by (4) is given as follows.

$$\begin{cases} x(k+1) = (A_0 + B_2K)x(k) + A_1x(k-d(k)) + B_1w(k) \\ z(k) = (C_0 + D_{12}K)x(k) + C_1x(k-d(k)) + D_{11}w(k) \\ x(k) = \phi(k),\ -\bar{h} \le k \le 0 \end{cases} \quad (10)$$

Applying Lemma 1 yields the following result.

Proposition 1: Given $\gamma > 0$, the system (10) is asymptotically stable with a prescribed H_∞ performance γ for any time-varying delay satisfying (3) if there exist matrices $P > 0, R > 0, Q > 0, M := [M_1\ M_2\ M_3], Z := (Z_{ij})_{3\times 3}$ with appropriate dimensions such that

$$\Omega_1 := \begin{bmatrix} \Phi & \Gamma_1^T & h\Gamma_1^T & \Gamma_2^T \\ * & -P^{-1} & 0 & 0 \\ * & * & -hR^{-1} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (11)$$

$$\Omega_2 := \begin{bmatrix} R & M \\ * & Z \end{bmatrix} \ge 0 \quad (12)$$

where

$$\Phi = \begin{bmatrix} \digamma & PA_1 - M_1^T + M_2 + \bar{h}Z_{12} & PB_1 + M_3 + \bar{h}Z_{13} \\ * & -M_2^T - M_2 + \bar{h}Z_{22} & -M_3 + \bar{h}Z_{23} \\ * & * & -\gamma^2 I + \bar{h}Z_{33} \end{bmatrix}$$

$$\digamma = PA_K + A_K^TP + M_1^T + M_1 + (\bar{h} - \underline{h} + 1)Q + \bar{h}Z_{11}$$
$$\Gamma_1 = [A_K\ \ A_1\ \ B_1]$$
$$\Gamma_2 = [C_0 + D_{12}K\ \ C_1\ \ D_{11}]$$
$$A_K = A_0 + B_2K - I$$

Proof: Choose a Lyapunov-Krasovskii functional candidate as

$$V(k) = V_1(k) + V_2(k)$$

where

$$V_1(k) = x^T(k)Px(k)+ \sum_{\theta=-\bar{h}+1}^{0} \sum_{j=k-1+\theta}^{k-1} y^T(j)Ry(j)$$

$$V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)+ \sum_{j=-\bar{h}+2}^{-\underline{h}+1} \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l)$$

where $P > 0, R > 0$,and $Q > 0$ are to be determined and $y(k)$ defined in (5). Taking the forward difference gives

$$\begin{aligned}\Delta V_1(k) =&V_1(k+1) - V_1(k)\\ =&2x^T(k)Py(k)+y^T(k)(P+hR)y(k)\\ &- \sum_{j=k-\bar{h}}^{k-1} y^T(j)Ry(j)\end{aligned} \tag{13}$$

From (10), we have

$$y(k) = A_K x(k)+A_1 x(k-d(k))+B_1 w(k) \tag{14}$$

In addition, from (3), it is easily deduced that

$$- \sum_{j=k-\bar{h}}^{k-1} y^T(j)Ry(j) \le - \sum_{j=k-d(k)}^{k-1} y^T(j)Ry(j) \tag{15}$$

Substituting (7) into (15), and together with (13) and (14) yields

$$\Delta V_1(k) \le \xi^T(k)[\Xi + \Gamma_1^T(P + \bar{h}R)\Gamma_1]\xi(k) \tag{16}$$

where Γ_1 is defined in (11) and

$$\Xi = \begin{bmatrix} PA_K+A_K^TP+\upsilon_{11} & PA_1+\upsilon_{12} & PB_1+M_3+\bar{h}Z_{13}\\ * & \upsilon_{22} & -M_3+\bar{h}Z_{23}\\ * & * & \bar{h}Z_{33}\end{bmatrix}$$

where $\upsilon_{11}, \upsilon_{12}, \upsilon_{22}$ are defined in (7).

Similar to [12], we obtain

$$\Delta V_2(k) \le (\bar{h}-\underline{h}+1)x^T(k)Qx(k)-x^T(k-d(k))Qx(k-d(k)). \tag{17}$$

Therefore, from (16) and (17), we have

$$\Delta V(k) \le \xi^T(k)[\Xi + \Xi_1 + \Gamma_1^T(P + \bar{h}R)\Gamma_1]\xi(k) \tag{18}$$

where $\Xi_1 = \mathrm{diag}\{(\bar{h}-\underline{h}+1)Q, -Q, 0\}$.

Now, we prove the conclusion from two aspects. First, we show that the system (10) with $w(k) = 0, \forall k \ge 0$, is asymptotically stable if (11) and (12) are satisfied. For this situation, (18) becomes

$$\Delta V(k) \le \tilde{\xi}^T(k)[\tilde{\Xi} + \tilde{\Gamma}_1^T(P + \bar{h}R)\tilde{\Gamma}_1]\tilde{\xi}(k) \tag{19}$$

where

$$\begin{aligned}\tilde{\xi}(k) &= \mathrm{col}\{x(k), x(k-d(k))\}\\ \tilde{\Gamma}_1 &= [A_K \;\; A_1]\\ \tilde{\Xi} &= \begin{bmatrix} PA_K+A_K^TP+\upsilon_{11}+(\bar{h}-\underline{h}+1)Q & PA_1+\upsilon_{12}\\ * & -Q+\upsilon_{22}\end{bmatrix}\end{aligned}$$

On the other hand, the matrix inequality (11) implies

$$\begin{bmatrix}\tilde{\Xi} & \tilde{\Gamma}_1^T & \bar{h}\tilde{\Gamma}_1^T\\ * & -P^{-1} & 0\\ * & * & -\bar{h}R^{-1}\end{bmatrix} < 0 \tag{20}$$

Applying Schur complement yields $\Delta V(k) < 0$ if (11) and (12) are true. Thus, we can conclude from the Lyapunov-Krasovskii stability theorem in [5] that the system (10) with $w(k) = 0, \forall k \ge 0$, is asymptotically stable.

Next, under zero initial condition, the system (10) has a prescribed H_∞ attenuation level γ, i.e. $\|z\|_2 < \|w\|_2$ for all $w(k) \ne 0$. To show this, we rewrite (18) as

$$\begin{aligned}&\Delta V(k)+z^T(k)z(k)-\gamma^2 w^T(k)w(k)\\ &\le \xi^T(k)[\Phi+\Gamma_1^T(P+\bar{h}R)\Gamma_1+\Gamma_2^T\Gamma_2]\xi(k)\end{aligned} \tag{21}$$

where Φ and Γ_2 are defined in (11). Clearly, if the matrix inequality (11) holds, using Schur complement gives

$$\Delta V(k)+z^T(k)z(k)-\gamma^2 w^T(k)w(k) < 0 \tag{22}$$

Taking the sum of two side of (22) from 0 to ∞ yields

$$\sum_{k=0}^{\infty}[z^T(k)z(k) - \gamma^2 w^T(k)w(k)] < V(0) - V(\infty)$$

Under zero initial condition, $V(0) = 0$, we have

$$\sum_{k=0}^{\infty}[z^T(k)z(k) - \gamma^2 w^T(k)w(k)] < 0$$

that is, $\|z\|_2 < \gamma\|w\|_2$, which completes the proof. □

Remark 3: From the proof of Proposition 1, it is clear to see that the new inequality (7) plays an important role in the derivation of the delay-dependent condition. With its help, neither model transformation nor bounding technique for cross terms is employed.

Clearly, the matrix inequality (11) is nonlinear due to the terms PA_K and A_K^TP etc. In order to solve the controller gain K from (11) and (12), we rewrite Proposition 1 as follows.

Proposition 2: Given $\gamma > 0$, the system (10) is asymptotically stable with a prescribed H_∞ performance γ for any time-varying delay satisfying (3) if there exist real matrices $\mathcal{P} > 0, \mathcal{R} > 0, \mathcal{Q} > 0, Y, N_1, N_2, N_3, \mathcal{Z} := (\mathcal{Z}_{ij})_{3\times 3}$ with appropriate dimensions such that

$$\Upsilon_1 := \begin{bmatrix}\Sigma & \Pi_1^T & \bar{h}\Pi_1^T & \Pi_2^T\\ * & -\mathcal{P} & 0 & 0\\ * & * & -\bar{h}\mathcal{R} & 0\\ * & * & * & -I\end{bmatrix} < 0 \tag{23}$$

$$\Upsilon_2 := \begin{bmatrix}\mathcal{P}\mathcal{R}^{-1}\mathcal{P} & N_1 & N_2 & N_3\\ * & \mathcal{Z}_{11} & \mathcal{Z}_{12} & \mathcal{Z}_{13}\\ * & * & \mathcal{Z}_{22} & \mathcal{Z}_{23}\\ * & * & * & \mathcal{Z}_{33}\end{bmatrix} \ge 0 \tag{24}$$

where

$$\Sigma = \begin{bmatrix}\beth & A_1\mathcal{P}-N_1^T+N_2+\bar{h}\mathcal{Z}_{12} & B_1+N_3+\bar{h}\mathcal{Z}_{13}\\ * & -N_2^T-N_2+\bar{h}\mathcal{Z}_{22} & -N_3+\bar{h}\mathcal{Z}_{23}\\ * & * & -\gamma^2 I+\bar{h}\mathcal{Z}_{33}\end{bmatrix}$$

$$\begin{aligned}\beth =&(A_0-I)\mathcal{P}+\mathcal{P}(A_0-I)^T+B_2Y+Y^TB_2^T\\ &+N_1^T+N_1+(\bar{h}-\underline{h}+1)\mathcal{Q}+\bar{h}\mathcal{Z}_{11}\\ \Pi_1 =&[(A_0-I)\mathcal{P} + B_2Y \;\;\; A_1\mathcal{P} \;\;\; B_1]\\ \Pi_2 =&[C_0\mathcal{P}+D_{12}Y \;\;\; C_1\mathcal{P} \;\;\; D_{11}]\end{aligned}$$

Moreover, a suitable controller gain is given by $K = Y\mathcal{P}^{-1}$.

Proof: See the full version of the paper.

If we set $\mathcal{P} = \mathcal{R}$, then matrix inequalities (23) and (24) become linear, in which case it is easy to get a minimum H_∞ performance $\gamma_{\min}$ for given delay bounds $\bar{h}$ and $\underline{h}$, or to get a maximum delay upper bound $\bar{h}$ for given γ and $\underline{h}$ by using a convex optimization algorithm. However, this setting leads to more conservative results. In order to derive much better results, similar to [8], we convert the non-convex feasibility problem formulated by (23) and (24) into a nonlinear minimization problem subject to LMIs. In the beginning, we introduce a new matrix variable, $S > 0$, such that $\mathcal{P}\mathcal{R}^{-1}\mathcal{P} \geq S$, then, we replace the matrix inequality (24) with

$$\begin{bmatrix} S & N_1 & N_2 & N_3 \\ * & \mathcal{Z}_{11} & \mathcal{Z}_{12} & \mathcal{Z}_{13} \\ * & * & \mathcal{Z}_{22} & \mathcal{Z}_{23} \\ * & * & * & \mathcal{Z}_{33} \end{bmatrix} \geq 0 \tag{25}$$

$$\mathcal{P}\mathcal{R}^{-1}\mathcal{P} \geq S \tag{26}$$

On the other hand, it is easily shown that $\mathcal{P}\mathcal{R}^{-1}\mathcal{P} \geq S$ is equivalent to

$$\begin{bmatrix} S^{-1} & \mathcal{P}^{-1} \\ * & \mathcal{R}^{-1} \end{bmatrix} \geq 0$$

Let $T = S^{-1}, L = \mathcal{P}^{-1}, J = \mathcal{R}^{-1}$. By employing the cone complementarity problem proposed by [3], the nonlinear minimization problem subject to LMIs can be formulated as follows.

$$\text{Minimize } \ \mathrm{Tr}(ST + \mathcal{P}L + \mathcal{R}J) \tag{27}$$

Subject to (23), (25) and

$$\left.\begin{array}{l} \begin{bmatrix} T & L \\ L & J \end{bmatrix} \geq 0, \quad \mathcal{P} > 0, \quad \mathcal{R} > 0, \quad \mathcal{Q} > 0 \\ \begin{bmatrix} S & I \\ I & T \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mathcal{P} & I \\ I & L \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mathcal{R} & I \\ I & J \end{bmatrix} \geq 0 \end{array}\right\} \tag{28}$$

If the obtained value of $\mathrm{Tr}(ST+\mathcal{P}L+\mathcal{R}J)$ is exactly equal to $3n$, then it is clear from Proposition 2 that system (10) is asymptotically stable with a prescribed H_∞ attenuation level γ via the memoryless controller (4) with $K = Y\mathcal{P}^{-1}$. Applying the linearization method ([3]), we can easily derive a suboptimal H_∞ performance $\gamma_{\min}$ for given delay bounds $\underline{h}$ and $\bar{h}$ or a suboptimal maximum delay upper bound $\bar{h}$ for given γ and $\underline{h}$ by an iterative algorithm given in the following.

Algorithm: Minimize γ for given delay bounds $\underline{h}$ and $\bar{h}$.

1) Choose a sufficiently large initial γ_{ini} such that (23), (25) and (28) are feasible. Set $\gamma_{so} = \gamma_{ini}$.
2) Find a feasible set $(S^0, \mathcal{P}^0, \mathcal{R}^0, T^0, L^0, J^0, \mathcal{Q}^0, N_j^0, \mathcal{Z}_{ij}^0 \ (i,j = 1,2,3))$ satisfying (23), (25) and (28). Set $l = 0$.
3) Solve the following LMI problem for the varibles $(S, \mathcal{P}, \mathcal{R}, T, L, J, \mathcal{Q}, N_j, \mathcal{Z}_{ij} \ (i,j = 1,2,3))$:

   Minimize $\mathrm{Tr}(S^lT+T^lS+\mathcal{P}^lL+L^l\mathcal{P}+\mathcal{R}^lJ+J^l\mathcal{R})$
   Subject to (23), (25) and (28).

   Set $S^{l+1} = S, \ T^{l+1} = T, \ \mathcal{P}^{l+1} = \mathcal{P},$
   $L^{l+1} = L, \ \mathcal{R}^{l+1} = \mathcal{R}, \ J^{l+1} = J.$
4) If matrix inequality (24) and

   $$|\mathrm{Tr}(S^lT+T^lS+\mathcal{P}^lL+L^l\mathcal{P}+\mathcal{R}^lJ+J^l\mathcal{R})-6n| < \varepsilon \tag{29}$$

   where ε is a prescribed sufficiently small positive number, are satisfied, then set $\gamma_{so} = \gamma_{ini}$ and decrease γ_{ini} to some extent and back to Step 2. If one of the conditions (24) and (29) is not satisfied within a specfied number of iterations, then exit, otherwise, set $l = l + 1$ and go to Step 3.

Remark 4: The proposed algorithm provides an approach to obtaining a suboptimal H_∞ performance for given bounds $\underline{h}$ and $\bar{h}$. It can be also used to derive a suboptimal delay upper bound $\bar{h}$ for given γ and $\underline{h}$. The algorithm takes the matrix inequality (24) as one of stopping criteria since our main aim is to find a feasible solution such that (23) and (24) are satisfied, on the other hand, it is very difficult to exactly obtain the minimum value, $6n$, of $\mathrm{Tr}(S^lT+T^lS+\mathcal{P}^lL+L^l\mathcal{P}+\mathcal{R}^lJ+J^l\mathcal{R})$. The example in the next section shows that this algorithm can achieve some satisfactory results.

In the sequel, we present a robust result for H_∞ control of system (1) with uncertainties. Combining Proposition 2 with Lemma 2.4 in [11] gives the following result.

Proposition 3: Given $\gamma > 0$, the system (1) is asymptotically stable with a prescribed H_∞ performance γ for any uncertainties satisfying (2) and any time-varying delay satisfying (3) if there exist real matrices $\mathcal{P} > 0, \mathcal{R} > 0, \mathcal{Q} > 0, Y, N_1, N_2, N_3, \mathcal{Z} := (\mathcal{Z}_{ij})_{3\times 3}$ with appropriate dimensions and a scalar $\epsilon > 0$ such that (24) and the following matrix inequality hold:

$$\begin{bmatrix} \Sigma & \Pi_1^T & h\Pi_1^T & \Pi_2^T & \epsilon\Pi_3^T & \Pi_4^T \\ * & -\mathcal{P} & 0 & 0 & \epsilon D_1 & 0 \\ * & * & -h\mathcal{R} & 0 & \epsilon h D_1 & 0 \\ * & * & * & -I & \epsilon D_2 & 0 \\ * & * & * & * & -\epsilon I & 0 \\ * & * & * & * & * & -\epsilon I \end{bmatrix} < 0 \tag{30}$$

where Σ, Π_1, Π_2 are defined in (23); and

$$\Pi_3 = [D_1^T \ \ 0 \ \ 0]$$
$$\Pi_4 = [E_1\mathcal{P} + E_4Y \ \ E_2\mathcal{P} \ \ E_3]$$

Moreover, a suitable controller gain is given by $K = Y\mathcal{P}^{-1}$.

The previous algorithm is also valid for Proposition 3 only if we replace (23) with (30) in the algorithm. It will be shown by a numerical example that the obtained results are of less conservatism.

IV. A Numerical Example

In this section, a numerical example is used to demonstrate the validity of the proposed method.

Example 1: Consider the system

$$\begin{cases} x(k+1) = (A_0+DF(k)E_1)x(k)+B_1w(k)+B_2u(k) \\ \qquad\qquad + (A_1 + DF(k)E_2)x(k - d(k)) \\ z(k) = C_0x(k) + D_{12}u(k) \\ x(k) = 0, \quad -\bar{h} \leq k \leq 0 \end{cases} \tag{31}$$

TABLE I

THE MAXIMUM DELAY BOUND $\bar{h}$ FOR SYSTEM (31)

Method	h	K
Lee & Kwon [7]	41	$[-0.6311\ \ -2.3615]$
Fridman & Shaked [4]	67	unprovided
Proposition 3	70	[-93.2010 -71.2670]

TABLE II

THE ACHIEVED MINIMUM H_∞ PERFORMANCES γ AND CORRESPONDING CONTROLLER GAIN K FOR $\bar{h} = 64$

γ	K	Number of Iterations
20	$[-31.9456\ \ -41.2442]$	154
18	$[-36.1621\ \ -48.5035]$	171
17	$[-39.0994\ \ -53.8247]$	186
16	$[-44.9680\ \ -62.3831]$	223
15.5	$[-46.4416\ \ -68.1845]$	235

where

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1.01 \end{bmatrix},\ A_1 = \begin{bmatrix} -0.02 & -0.005 \\ 0 & -0.01 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$
$$C_0 = [1\ \ 0],\quad D_{12} = 0.1,\quad D = 0.02I$$
$$E_1 = E_2 = 0.01I,\quad F^T(k)F(k) \leq I.$$

and $d(k)$ is a delay satisfying (3).

In the following, two cases of delay $d(k)$ are considered.

Case 1: delay $d(k)$ is time-invariant, i.e. $\underline{h} = \bar{h}$.

In this case, we calculated the maximum delay bound $\bar{h}$, which can ensure that system (31) is asymptotically stable via memoryless state feedback (4). The obtained results are listed in Table I.

Moreover, Fridman and Shaked [4] also calculated the achieved minimum H_∞ performance and $\gamma_{\min} = 180.07$ was obtained for $\bar{h} = 64$. However, applying Proposition 3 combined with the iterative algorithm yields much less conservative results, which are listed in Table II. Note from the table that the proposed method provides much less H_∞ performance, $\gamma = 15.5$, than [4] for the same delay upper bound $\bar{h}$.

Case 2: delay $d(k)$ is time-varying satisfying $\underline{h} \leq d(k) \leq \bar{h}$.

In this case, the proposed conditions in [6], [10] incorporating with Lemma 2.4 in [11] are infeasible to this example. It is concluded from [4] that system (31) is robustly stabilizable for all $\bar{h} \leq 43$. When $\bar{h} = 43$, the system achieved the minimum H_∞ performance $\gamma_{\min} = 169.4722$ via memoryless state feedback (4) with $K = [-6.7766\ \ -20.5924]$. However, from Proposition 3, the system (31) is robust stabilizable for $\bar{h} \leq 48$. When $\bar{h} = 48$, different γ values, much less than 169.4722, are achieved for different $\underline{h}$, which are listed in Table III.

Clearly, the above results obtained by Proposition 3 are much less conservative than those in [4], [6], [10], which clearly shows the effectiveness of the proposed method in this paper.

TABLE III

THE ACHIEVED MINIMUM H_∞ PERFORMANCES γ AND CORRESPONDING CONTROLLER GAIN K FOR DIFFERENT $\underline{h}$ WHEN $\bar{h} = 48$

$\underline{h}$	γ	K
1	65	$[-24.8606\ \ -74.4157]$
8	50	$[-24.9197\ \ -76.6840]$
18	40	$[-22.2636\ \ -68.8382]$
28	30	$[-18.0179\ \ -57.7737]$
38	20	$[-14.5769\ \ -50.1003]$
43	18	$[-8.6551\ \ -36.0802]$

V. CONCLUSION

This paper discussed the robust H_∞ control problem for discrete-time linear systems with both time-varying delays and uncertainties. To solve the robust H_∞ control problem, a new *Finite Sum Inequality* based on quadratic terms is first established. Then, with its help, a less conservative delay-dependent criterion has been derived, under which the resulting closed-loop system can achieve a prescribed H_∞ performance. In addition, an iterative algorithm has been proposed to design a suboptimal H_∞ controller. A numerical example has been finally given to demonstrate the effectiveness of the proposed method.

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